A canonical martingale coupling

Workshop on Optimal Transportation and Appplications

Nicolas JUILLET

Université de Strasbourg

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Outline

Definition: martingale transport plan

A probability measure P on $\mathbb{R} \times \mathbb{R}$ is termed a martingale transport plan if $P = \mathrm{Law}(X, Y)$ where (X, Y) is a two-times martingale process.

Equivalently if $(P_x)_{x \in \mathbb{R}}$ is a disintegration (allias conditional laws, allias Markov kernel) of P, it has to satisfy

$$\mathrm{Barycenter}(P_x) = \int y \, \mathrm{d}P_x(y) = x$$

for $(\mathrm{proj}^x_\# P)$-almost every x.

Some examples

- $P = \text{Law}(x, Y)$ where $x = \mathbb{E}(Y)$.
- $P = \sum_{i=1}^{2}\sum_{j=1}^{3} a_{i,j}\,\delta_{(x_i,y_j)}$ where $x \in \{-1,1\}$ and $y \in \{-2,0,2\}$ and
$$(a_{i,j}) = \begin{pmatrix} 1/4 & 1/4 & 0 \\ 1/12 & 1/12 & 1/3 \end{pmatrix} + t\begin{pmatrix} 1/12 & -1/6 & 1/12 \\ -1/12 & 1/6 & -1/12 \end{pmatrix}$$
for some $t \in [0,1]$.
- $P = \text{Law}(X, X+I)$ where the increment I is independent from X. (for instance X and I are Gaussian)
- $P = \frac{P_1+P_2}{2}$ where P_1, P_2 are martingale transport plans.
- $P = \text{Law}(\mathbb{E}(Y|\mathcal{F}), Y)$ for some $\mathcal{F} \subseteq \sigma(Y)$.

The general problem

The problem

Minimize

$$P \mapsto \int c(x,y)\mathrm{d}P(x,y)$$

among the martingale transport plans from μ to ν.

For different cost functions c we would like to know:

- How do the minimizers look like?
- What are their properties?
- Is there a unique minimizer?

Model theorem

Model theorem in the classical setting

For μ and ν in $\mathcal{P}_2$ in the convex order and P a transport plan from μ to ν. The following statements are equivalent:

- The plan P is optimal for the transport problem with $c(x,y) = (y-x)^2$,
- The plan P is concentrated on a monotone set Γ,
- The plan P is the quantile coupling.

We have proved a theorem similar to this one in the martingale setting.

The convex order

Definition: the convex order

We write

$$\mu \preceq_C \nu$$

and say that μ is smaller than ν in the convex order if and only if there exists a martingale transport plan P with

$$\mathrm{proj}^x_\# P = \mu \quad \text{and} \quad \mathrm{proj}^y_\# P = \nu.$$

According to a (non constructive) theorem of Strassen, it is equivalent to assume

$$\int \varphi \, \mathrm{d}\mu \leq \int \varphi \, \mathrm{d}\nu$$

for every convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

The extended order and the shadows

Proposition - Definition

- We write $\mu \preceq_E \nu$ and say that μ is smaller than ν in the extended order if

$$F_\mu^\nu := \{\theta : \mu \preceq_C \theta \text{ and } \theta \leq \nu\}$$

  is not empty.
- The partially ordered set $(F_\mu^\nu, \preceq_C)$ has a minimum. We call it the shadow of μ in ν and denote it by $S^\nu(\mu)$.

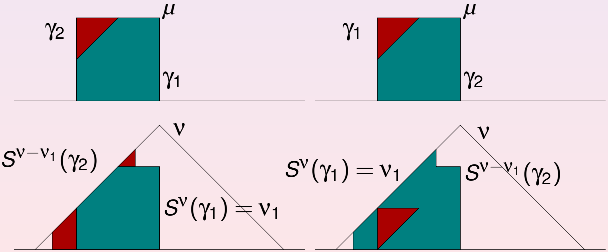

Figure: Shadow of μ in ν and associativity of the shadow projection.

The variational lemma

This lemma is a kind of *c*-cyclical monotonicity lemma for the martingale setting.

Variational Lemma

Let P be optimal, there exists $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ such that for any finitely supported measure α with $\alpha(\Gamma) = 1$, the minimum of $\alpha' \mapsto \int c(x,y)\,\mathrm{d}\alpha'(x,y)$ over

$$\mathrm{Competitor}(\alpha) = \left\{ \alpha' : \alpha' \begin{array}{l} \text{has the same marginals as } \alpha \\ \forall x \in \mathbb{R}, \int y\,\mathrm{d}\alpha_x = \int y\,\mathrm{d}\alpha'_x \end{array} \right\}$$

is obtained in α.

Martingale theorem

Theorem

For μ and ν in $\mathcal{P}_3$ in the convex order and P a martingale transport plan from μ to ν. The following statements are equivalent:

- The plan P is optimal for the martingale transport problem with cost $c(x,y) = (y-x)^3$,
- The plan P is concentrated on a martingale-monotone set Γ (*see the figure*),
- The plan P is the left- curtain coupling (*i.e., transports* $\mu_{]-\infty,x]}$ *to its shadow*)

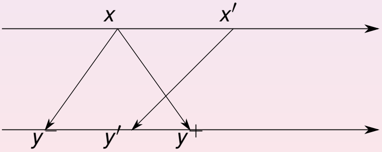

Figure: This configuration of three points (x,y), (x',y^-) and (x',y^+) is forbidden on martingale-monotone sets Γ.

One example

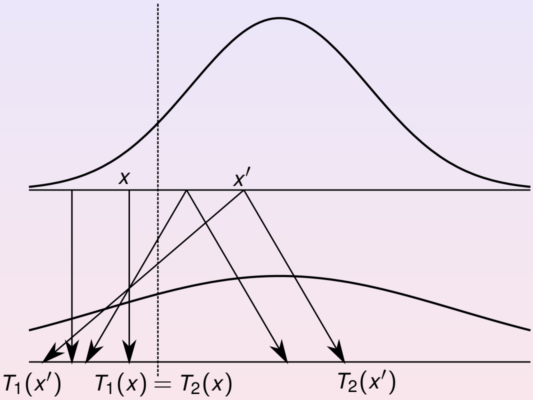

Figure: Optimal transport plan between Gaussian measures.

Corollary

Corollary for μ continuous

If μ is continuous (=no atom), there are $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ such that the optimal P is concentrated on $\mathrm{graph}(T_1) \cup \mathrm{graph}(T_2)$.

The variational lemma is of general use, especially when μ is continuous.

Examples

- $c(x,y) = -|y-x|$
- $c(x,y) = |y-x|$
- $c(x,y) = (y-x)^n$