Diffusion by optimal transport in the Heisenberg group

Perspectives in Optimal transportation

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The geometry of $\mathbb{H}$.

- A basis for left-invariant vector fields is

$$\mathbf{X} = E_1 - \frac{y}{2}E_3 \quad , \quad \mathbf{Y} = E_2 + \frac{x}{2}E_3 \quad \text{and} \quad \mathbf{U} = [\mathbf{X}, \mathbf{Y}] = E_3.$$

- A curve is horizontal if $\dot{\gamma} \in \mathrm{Vect}(\mathbf{X}, \mathbf{Y})$ for any t. Actually

$$\dot{\gamma}(t) = a(t)\mathbf{X}(\gamma(t)) + b(t)\mathbf{Y}(\gamma(t)).$$

  It has norm $|\dot{\gamma}| = \sqrt{a^2(t) + b^2(t)}$.

- The diffusion operator is $\Delta = \mathbf{X}^2 + \mathbf{Y}^2$.

The Riemannian Heisenberg group $\mathbb{H}_\varepsilon$

Let $\varepsilon > 0$ and $\mathbb{H}_\varepsilon := (\mathbb{R}^3, d_\varepsilon, \mathcal{L}^3)$.

- An orthonormal basis at point (x, y, u) is

$$\mathbf{X} = E_1 - \frac{y}{2}E_3 \quad , \quad \mathbf{Y} = E_2 + \frac{x}{2}E_3 \quad \text{and} \quad \varepsilon\mathbf{U} = \varepsilon[\mathbf{X}, \mathbf{Y}] = \varepsilon E_3.$$

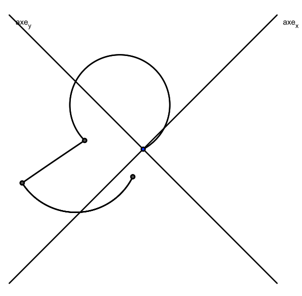
- If

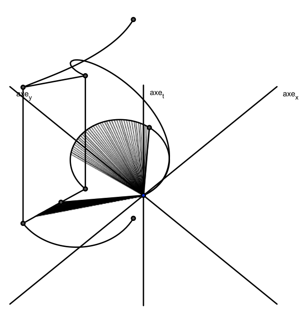
$$\dot\gamma(t) = (a(t)\mathbf{X} + b(t)\mathbf{Y} + c(t)\mathbf{U})(\gamma(t))$$

  then $|\dot\gamma(t)|_\varepsilon = \sqrt{a^2 + b^2 + \frac{c^2}{\varepsilon^2}}$.

- The diffusion operator is $\Delta_\varepsilon = \Delta + \varepsilon^2\mathbf{U}^2$.

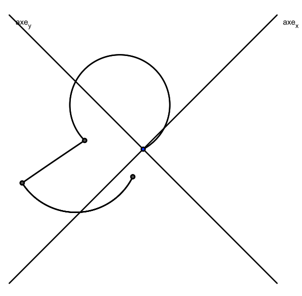
Geodesics of $\mathbb{H}$.

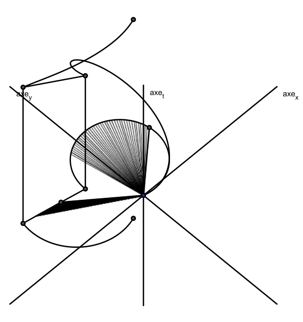




- A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.
- The length of the horizontal curves is exactly the length of the projection in $\mathbb{C}$.
- The geodesics of $\mathbb{H}$ are the horizontal curves whose projection is an arc of circle or a line.

Geodesics of $\mathbb{H}$.





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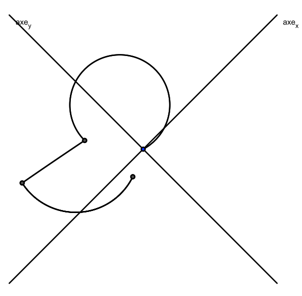

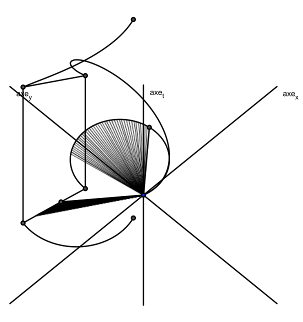

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Wasserstein metrics

Let W be the L^2-minimal metric with respect to the Carnot-Carathéodory metric d and W_ε with respect to d_ε.

- $W_\varepsilon \leq W$,
- $\mathcal{P}_2(\mathbb{H}) = \mathcal{P}_2(\mathbb{H}_\varepsilon)$ as topological spaces.
- Lipschitz curves (resp. absolutely continuous) of $\mathcal{P}_2(\mathbb{H})$ are Lipschitz (resp. absolutely continuous) in $\mathcal{P}_2(\mathbb{H}_\varepsilon)$.

The relative entropy

The relative entropy of $\mu = \rho\mathcal{L}$ is given by

$$H(\mu) = H(\mu \mid \mathcal{L}) = \int \rho \ln(\rho)(x) d\mathcal{L}(x).$$

Big entropy: μ concentrated on a few space.
Small entropy: μ take a lot of space.

Synthetic Ricci curvature – definition

A space (X, d, ν) satisfies $CD(0, +\infty)$ if

$\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$ absolutely continuous, there exists a geodesic $(\mu_t)_{t\in[0,1]}$ such that $t \in [0,1] \to \mathrm{Ent}(\mu_t \mid \nu) \in \mathbb{R}$ is convex.

For $\mathbb{H}_\varepsilon = (\mathbb{R}^3, d_\varepsilon, \mathcal{L}^3)$, the curvature-dimension condition $CD(-\frac{1}{2\varepsilon^2}, 3)$ is satisfied.

Theorem (J.)

For $\mathbb{H} = (\mathbb{R}^3, d_c, \mathcal{L}^3)$, the curvature-dimension condition $CD(K, N)$ is not satisfied for any $N \in [1, +\infty[$ and $K \in \mathbb{R}$.

Gradient flow of the relative entropy.

An absolutely continuous curve $(\mu_t)_{t\geq 0}$ is a gradient flow of H if

- $t \mapsto H(\mu_t)$ is decreasing.
-
$$\text{for almost every } t > 0, \quad \begin{cases} \partial_t H(\mu_t) = -\,\text{Speed}(\mu_t) \cdot \text{Slope}(H)(\mu_t) \\ \text{Speed}(\mu_t) = \text{Slope}(H)(\mu_t). \end{cases}$$

Theorem (J.)

The curve $(\mu_t)_{t>0}$ is a gradient flow of H if and only if $\mu_t = \rho_t \mathcal{L}^3$ with

$$\frac{\partial}{\partial t}\rho_t = \Delta \rho_t.$$

HWI inequalities

From

$$H(\mu) \geq H(\nu) - \sqrt{I_\varepsilon(\nu)}W_\varepsilon(\mu,\nu) + \frac{1}{2} \cdot \left(\frac{-1}{2\varepsilon^2}\right) W_\varepsilon(\mu,\nu)^2$$

to

$$H(\mu) \geq H(\nu) - \sqrt{I(\nu)}W(\mu,\nu) - C(\mu)W(\mu,\nu)^{3/2},$$

but only if $I_\varepsilon(\nu) = I(\nu) + \varepsilon J(\nu)$ is finite.

Strong upper gradient property for H on $\mathcal{P}_2(\mathbb{H}_\varepsilon)$ – definition

Let $(\mu_t)_{t\geq 0}$ be an absolutely continuous curve of $\mathcal{P}_2(\mathbb{H}_\varepsilon)$.

If

$$\int_{t_0}^{t_1} \mathrm{Slope}_\varepsilon(H)(\mu_t)\,\mathrm{Speed}_\varepsilon(\mu_t) < +\infty$$

then $t \mapsto H(\mu_t)$ is absolutely continuous.