

# About Ricci curvature in the sub-Riemannian Heisenberg group

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# Outline

- 1 Optimal transport and geodesic interpolation
- 2 The first Heisenberg group
- 3 The contraction estimate and two results

## Geodesic space

In a metric space  $(X, d)$ ,  $\gamma: [0, 1] \rightarrow X$  is a geodesic if for every  $s, s' \in [0, 1]$

$$d(\gamma(s), \gamma(s')) = |s - s'|d(\gamma(0), \gamma(1)).$$

A metric space  $(X, d)$  is geodesic if for all  $(p, q)$  there is a geodesic  $\gamma$  from  $p$  to  $q$ .

If  $\gamma$  is unique, we define  $\mathcal{M}^s(p, q) = \gamma(s)$ , the interpolation map.

# Rényi Entropy

The  $N$ -entropy of a probability measure  $\mu$  of density  $\rho$  is given by

$$\text{Ent}_N(\rho \nu \mid \nu) = - \int \rho^{1-1/N} d\nu(x).$$

If  $\mu$  is singular  $\text{Ent}(\mu) = 0$ .

Big entropy:  $\mu$  concentrated on a small space.

Small entropy:  $\mu$  fills a lot of space.

## Curvature-dimension $CD(0, N)$

A space  $(X, d, \nu)$  satisfies  $CD(0, N)$  if

for every absolutely continuous  $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ , there is a geodesic  $(\mu_s)_{s \in [0,1]}$  such that  $s \in [0, 1] \rightarrow \text{Ent}_N(\mu_s | \nu) \in \mathbb{R}$  is convex.

(The exact statement is: there exists a geodesic  $(\mu_s)_{s \in [0,1]}$  such that for any  $s \in [0, 1]$ ,  $\text{Ent}_N(\mu_s | \nu) \leq (1 - s)\text{Ent}_N(\mu_0 | \nu) + s\text{Ent}_N(\mu_1 | \nu)$ )

## Synthetic Ricci curvature

New definitions of positive Ricci curvature for metric measure spaces  $(X, d, \nu)$ :

- Measure Contraction Property  $MCP(K, N)$  (Sturm; Ohta 2006)
- Curvature-Dimension  $CD(K, N)$  (Lott-Villani; Sturm 2006)

$$CD(0, N) \Rightarrow \text{Brunn-Minkowski}(0, N) \Rightarrow MCP(0, N).$$

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The Heisenberg group  $\mathbb{H}$  is  $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$  with the multiplicative structure:

$$(x, y, t) \cdot (x', y', t') = (z, t) \cdot (z', t') = \left( z + z', t + t' - \frac{\operatorname{Im}(z\bar{z}')}{2} \right).$$

The Lebesgue measure  $\mathcal{L}^3$  is left-invariant.

The left invariant vector fields

$$\mathbf{X} = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{Y} = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial t}$$

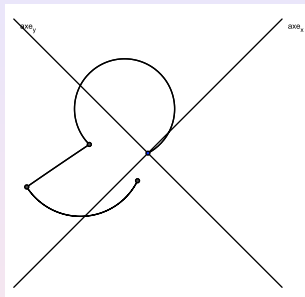
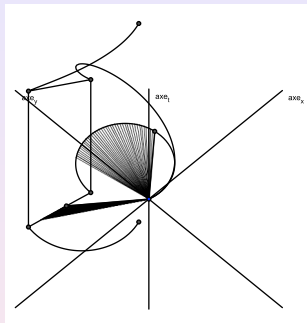
and  $\mathbf{T} = [\mathbf{X}, \mathbf{Y}] = \frac{\partial}{\partial t}$  span the tangent space in any point.

$$d_c(p, q) = \inf_{\gamma} \int_0^1 \sqrt{a^2(s) + b^2(s)} ds$$

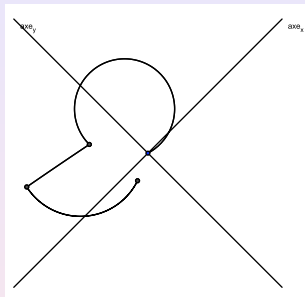
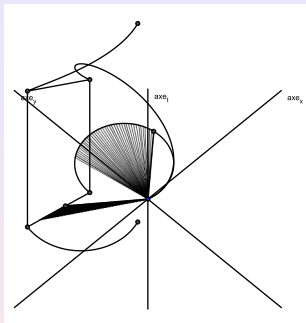
where  $\gamma$  is horizontal:

$$\dot{\gamma}(s) = a(s)\mathbf{X}(\gamma(s)) + b(s)\mathbf{Y}(\gamma(s)).$$

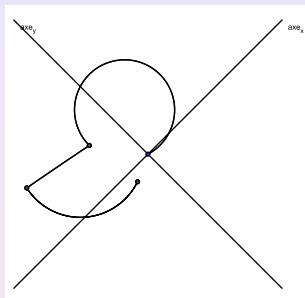
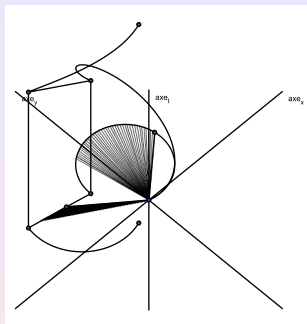


Geodesics of  $\mathbb{H}_1$ .

- A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.
- The length of the horizontal curves is exactly the length of the projection in  $\mathbb{C}$ .
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## The key estimate

### Key estimate

For any  $e \in \mathbb{H}$ . The contraction map  $\mathcal{M}_e^s : f \rightarrow \mathcal{M}^s(e, f)$  is differentiable with

$$\text{Jac}(\mathcal{M}_e^s)(f) \geq s^5.$$

Equality case :  $e$  and  $f$  are on a line.

As a consequence  $(\mathbb{H}, d_c, \mathcal{L}^3)$  satisfies  $MCP(0, 5)$ :

(Rough) definition of the Measure Contraction Property  $MCP(0, N)$  for  $(X, d, \nu)$ :

for every point  $e \in X$ , for every  $F \subset X$  and for all  $s \in [0, 1]$ ,

$$\nu(\mathcal{M}^s(e, F)) \geq s^N \nu(F).$$

## First result

### Theorem (Ambrosio, Rigot, 2004)

Let  $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H})$  such that  $\mu_0$  is absolutely continuous. Then there is a **unique** optimal coupling  $\pi$ . It is  $\pi = (\text{Id} \otimes T)_{\#} \mu_0$  for some map  $T$ . Moreover there is a **unique** geodesic between  $p$  and  $T(p)$  ( $\mu_0$ -almost surely).

Let  $T_s(p) = \mathcal{M}^s(p, T(p))$ . There is a unique geodesic  $(\mu_s)_{s \in [0,1]}$  between  $\mu_0$  and  $\mu_1$ .

It is defined by  $\mu_s = \mathcal{M}^s(\mu_0, \mu_1) = (T_s)_{\#} \mu_0$ .

### Open question (Ambrosio, Rigot)

Let  $\mu_0$  be absolutely continuous and  $s < 1$ . Is  $\mu_s$  absolutely continuous as well?

### Theorem (Figalli, J.)

Let  $(\mu_s)_{s \in [0,1]}$  be a geodesic of  $\mathcal{P}_2(\mathbb{H})$  and  $\mu_0$  absolutely continuous with respect to  $\mathcal{L}^3$ . Then for all  $s \in [0, 1)$ ,  $\mu_s$  is absolutely continuous too.

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## Second result

### Theorem (J.)

*In  $(\mathbb{H}, \mathcal{L}^3, d_C)$ , the Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance*

- *MCP(0, N) is true if and only if  $N \geq 5$ ,*
- *CD(0, N) and BM(0, N) are false for every N.*

### Theorem (J.)

*In  $(\mathbb{H}_n, \mathcal{L}^{2n+1}, d_C)$ , the  $n$ th-Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance*

- *MCP(K, N) is true if and only if  $N \geq 2n + 3$  and  $K \leq 0$ ,*
- *CD(K, N) and BM(K, N) are false for every  $(K, N)$ .*



## Brunn-Minkowski inequality $BM(0, N)$ :

A space  $(X, d, \nu)$  satisfies the Brunn-Minkowski inequality  $BM(0, N)$  if

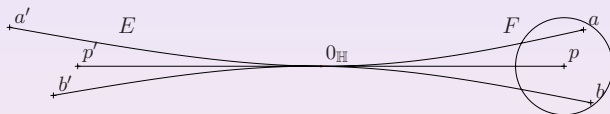
For every  $E, F \subset X$  and for all  $s \in [0, 1]$ ,

$$\nu(\mathcal{M}^s(E, F))^{1/N} \geq (1-s)\nu(E)^{1/N} + s\nu(F)^{1/N}.$$

In particular if  $\nu(E) = \nu(F)$ ,

$$\nu(\mathcal{M}^{1/2}(E, F))^{1/N} \geq \frac{\nu(E)^{1/N} + \nu(F)^{1/N}}{2} = \nu(E)^{1/N} = \nu(F)^{1/N}.$$

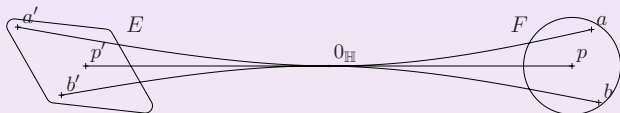
## Sketch of proof



Let  $F$  be a small ball such that  $0_{\mathbb{H}}$  and the center of the ball are on a “bad” geodesic.

For  $E$  we take the “geodesic inverse” of  $F$ .

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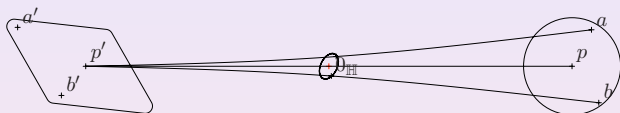


Let  $F$  be a small ball such that  $0_{\mathbb{H}}$  and the center of the ball are on a “bad” geodesic.

For  $E$  we take the “geodesic inverse” of  $F$ . It turns out that  $\mathcal{L}^3(E) = \mathcal{L}^3(F)$ .

We want to prove  $\mathcal{L}^3(\mathcal{M}^{1/2}(E, F)) < \mathcal{L}^3(F)$  because it is a contradiction to the  $BM(0, N)$ .

## Sketch of proof

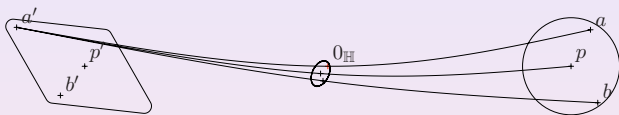


For each  $e \in E$ , the contracted set  $\mathcal{M}^{1/2}(e, F)$  is a sort of ellipsoid that contains  $0_{\mathbb{H}}$ .

The volume of such an ellipsoid is

$$2^{-5} \mathcal{L}^3(F).$$

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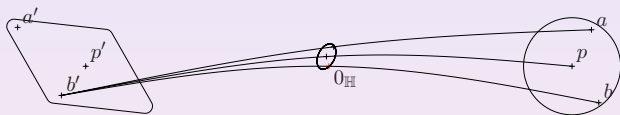


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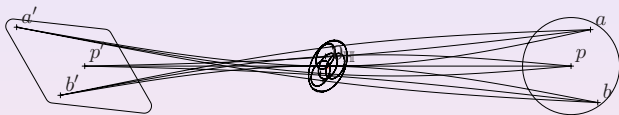


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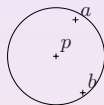
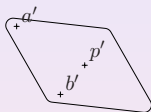
$$2^{-5} \mathcal{L}^3(F).$$

## Sketch of proof



The midset  $\mathcal{M}^{1/2}(E, F)$  is made of the union of these ellipsoids.

## Sketch of proof



All of them contain  $0_{\mathbb{H}}$ . Then  $\mathcal{M}^{1/2}(E, F)$  is an ellipsoid of size 2. Its volume is

$$2^3 (\cdot 2^{-5} \mathcal{L}^3(F)) = \frac{\mathcal{L}^3(F)}{4}.$$

Then  $\mathcal{L}^3(\mathcal{M}^{1/2}(E, F)) < \mathcal{L}^3(F) = \mathcal{L}^3(E)$ .