# A COUNTEREXAMPLE FOR THE GEOMETRIC TRAVELING SALESMAN PROBLEM IN THE HEISENBERG GROUP

NICOLAS JUILLET

Abstract. We are interested in characterizing the compact sets of the Heisenberg group that are contained in a curve of finite length. Ferrari, Franchi and Pajot recently gave a sufficient condition for those sets, adapting a necessary and sufficient condition due to P. Jones in the Euclidean setting. We prove that this condition is not necessary.

## Introduction

In the Euclidean setting a subset $E \subset \mathbb{R}^n$ is said to be $d$-rectifiable if there exists a countable family of Lipschitz maps $(f_k)_{k\in\mathbb{N}}$ from $\mathbb{R}^d$ to $\mathbb{R}^n$ such that $\mathcal{H}^d\left(E \setminus (\bigcup_{k=1}^{+\infty} f_k(\mathbb{R}^d))\right) = 0$, where $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure. Federer generalized this notion in [4] with the $d$-rectifiable metric spaces $(X, \rho)$. These spaces have to be covered, up to a set of $\mathcal{H}^d_\rho$-measure 0 by $\bigcup_{k=1}^{+\infty} f_k(U_k)$ where $f_k$ is Lipschitz, $U_k \subset \mathbb{R}^d$ and $\mathcal{H}^d_\rho$ is the $d$-dimensional Hausdorff measure with respect to distance $\rho$. Unfortunately, as observed by Ambrosio and Kirchheim [1], in the Heisenberg group with its Carnot-Carathéodory distance $(\mathbb{H}, d_c)$ this definition does not make much sense for dimensions $d \geq 2$. Indeed for these dimensions, $d$-rectifiable metric spaces included in $(\mathbb{H}, d_c)$ have vanishing $d$-Hausdorff measure. It should be noticed that a definition of rectifiable set in codimension 1 has been proposed by Franchi, Serapioni and Serra-Cassano [7, 8] in connection with sets of finite perimeter and BV functions. The case $d = 1$ is particular. Indeed, any rectifiable curve in a metric space can be parametrized by arclength and is the Lipschitz image of an interval of $\mathbb{R}$. Hence they are a lot of non-trivial 1-rectifiable metric spaces included in $(\mathbb{H}, d_c)$.

A more quantitative study of rectifiability properties of subsets of the complex plane has been introduced by P. Jones in connection with problems in harmonic analysis and complex analysis ($L^2$-boundness of the Cauchy operator on Lipschitz graphs, geometric characterisation of removable sets for bounded analytic functions in $\mathbb{C}$). This study has been pursued by David and Semmes in general spaces and has led to the notion of uniform rectifiability [3]. From the work of P. Jones arises the following problem that is known now as the geometric traveling salesman problem or analyst's traveling salesman problem: under which condition a compact set $E$ in a metric space $(X, \rho)$ is contained in a rectifiable curve? In the complex plane, P. Jones gives a complete characterisation of such sets by introducing $\beta$ numbers. These quantities measure how well the set $E$ is approximated by straight lines at each scale and each place.

In [5] Ferrari, Franchi and Pajot adapted the $\beta$ number of P. Jones to $\mathbb{H}$ and proved that a condition similar to that of P. Jones is sufficient for being contained in a rectifiable curve. In this paper, we prove that this condition is not necessary. Our counterexample is a curve $\omega[0,1] \subset \mathbb{H}$ of finite length. This curve is constructed in an iterative way and Figure 1 represented the projection on $\mathbb{C}$ of the first three curves (where $\mathbb{H}$ is seen as $\mathbb{C}\times\mathbb{R}$). The construction is inspired by the construction of the classical Koch snowflake.

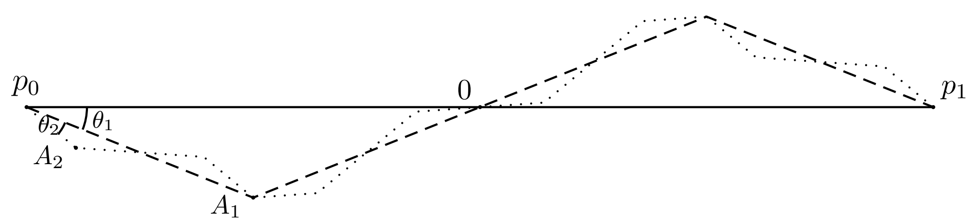


FIGURE 1. The counterexample curve.

0.1. **Definitions.** In order to give the characterization of P. Jones (and then of Ferrari, Franchi and Pajot), we must first define what is a dyadic net of a compact subset $E$ in a metric space $(X,\rho)$. It is an increasing sequence $(\Delta_k)_{k\in\mathbb{Z}}$ of subsets of $E$ such that for any $k\in\mathbb{Z}$,

- for all $x_1, x_2 \in \Delta_k$, we have $x_1 = x_2$ or $\rho(x_1,x_2) > 2^{-k}$,
- for any $y \in E$ there exists $x \in \Delta_k$ such that $\rho(y,x) \leq 2^{-k}$.

Actually for any compact set $E$, there exists such a dyadic net $(\Delta_k)_{k\in\mathbb{Z}}$. In this paper the results are independent of the choice of the dyadic net. We define

$$B_X^{\Delta}(E) = \sum_{k\in\mathbb{Z}} 2^{-k} \sum_{x\in\Delta_k} \beta_X^2(x, A\cdot 2^{-k})(E) \tag{1}$$

where $A>1$ is a constant to be specified (we will only assume that it is greater than 5) and $\beta_X(x,r)(E)$ depends on the ambient space. In th Euclidean case,

$$\beta_{\mathbb{R}^n}(x,r)(E) = \min_{l \text{ is a line}} \frac{\max_{y\in E\cap\mathcal{B}(x,r)} \rho(y,l)}{r}.$$

We consider actually the maximum distance to Euclidean lines $l$ of the points of $E$ that are included in $\mathcal{B}(x,r)$. The minimum of this quantity over $l$ is $\beta_{\mathbb{R}^n}(x,r)(E)$. A set that is "flat" around $x$ at scale $r$ will have a small $\beta$ number. We give a version of P. Jones' theorem as it is formulated in the survey [16]. The original theorem is given for dyadic squares instead of a dyadic net. Moreover the result proved in $\mathbb{R}^2$ by P. Jones [12] has actually been generalized by Okikiolu [15] who gave the reverse implication for the Euclidean spaces of greater dimensions.

**Theorem 0.1** ([12, 15]). *There exists a constant $C > 0$ (independent of the dyadic net $\Delta$) such that for any compact subset $E \subset \mathbb{R}^n$ with $B^{\Delta}_{\mathbb{R}^n}(E) < +\infty$, there are Lipschitz curves $\Gamma = \gamma([0,1]) \supset E$ satisfying the following inequality*

$$\mathcal{H}^1(\Gamma) \leq C \left(\operatorname{diam}(E) + B^{\Delta}_{\mathbb{R}^n}(E)\right)$$

*and for each of these curves*

$$B^{\Delta}_{\mathbb{R}^n}(E) \leq C\mathcal{H}^1(\Gamma).$$

In [17], Schul proved that in the previous result, one can find a constant $C$ that is independent to the dimension $n$ while it was not the case in the original proof of Theorem 0.1, where the $\beta$ numbers are taken on dyadic squares ($C$ depends exponentially on the dimension). It permitted him to prove a similar theorem for separable Hilbert spaces. From there it is natural to try to prove the same type of result in other metric spaces. In general metric spaces $(X, \rho)$ there is an article by Haolama [10] where the author uses the Menger curvature in the definition of the $\beta_X$ numbers. There is namely no definitely good definition of lines in $(X, \rho)$ for the geometric traveling salesman problem. In the case of the first Heisenberg group $\mathbb{H}$, Ferrari, Franchi and Pajot [5] obtain the exact counterpart of the beginning of Theorem 0.1 by using $\mathbb{H}$-lines (see Subsection 1.4) in the definition of $\beta_{\mathbb{H}}(x,r)$. Precisely

$$\beta_{\mathbb{H}}(x,r)(E) = \min_{l \text{ is a } \mathbb{H}\text{-line}} \frac{\max_{y \in E \cap \mathcal{B}^{\mathbb{H}}(x,r)} d_c(y,l)}{r}$$

where the balls $\mathcal{B}^{\mathbb{H}}(x,r)$ are the balls of $\mathbb{H}$. It is observed in [5] that the $\mathbb{H}$-lines are the left-translations $\tau_p(l_0)$ of the lines $l_0$ going through $0_{\mathbb{H}}$ in the plane $\mathbb{C} \times \{0_{\mathbb{R}}\}$, that is the $\mathbb{H}$-line going through $0_{\mathbb{H}}$.

The authors show that if the quantity $B^{\Delta}_{\mathbb{H}}(E)$ of (1) is finite, there is a rectifiable curve $\gamma$ covering $E$. Note that as a rectifiable curve, $\gamma$ has a Lipschitz parametrization on $[0,1]$. We give here a discrete version of the theorem – in the original theorem $B_{\mathbb{H}}$ is defined by integrating continuously the $\beta^2_{\mathbb{H}}$ on $\mathbb{H} \times \mathbb{R}^+$.

**Theorem 0.2** ([5]). *Let $E$ be a compact subset of $\mathbb{H}$ and $\Delta$ a dyadic net. Then if $B^{\Delta}_{\mathbb{H}}(E) < +\infty$ there is a Lipschitz curve $\Gamma = \gamma([0,1])$ such that $E \subset \Gamma$. Moreover, $\Gamma$ can satisfy*

$$\mathcal{H}^1(\Gamma) \leq C \left(\operatorname{diam}(E) + B^{\Delta}_{\mathbb{H}}(E)\right)$$

*where the constant $C$ is independent of $E$ and of its the dyadic net.*

They also prove that for regular enough curves of finite length, $B^{\Delta}_{\mathbb{H}}$ is finite.

**Proposition 0.3** ([5]). *Let $\gamma : [0,1] \to \mathbb{H}$ be $\mathcal{C}^{1,\alpha}$-curve, i.e. a horizontal curve such that the projection on $\mathbb{C}$, $Z(\gamma)$ is a $\mathcal{C}^{1,\alpha}$-curve. Then*

$$B^{\Delta}_{\mathbb{H}}(\gamma([0,1])) < +\infty.$$

The previous theorem suggests that it should be possible to characterize any compact set $E$ contained in a rectifiable curve by the condition $B_{\mathbb{H}}(E) < +\infty$. This would in particular happen for rectifiable curves themselves. Our curve $\omega([0,1])$ is a counterexample to this statement.

**Theorem 0.4.** *There is a Lipschitz curve $\omega : [0,1] \to \mathbb{H}$ such that for any dyadic net $\Delta$ of the set $\Omega = \omega([0,1])$,*

$$B^{\Delta}_{\mathbb{H}}(\Omega) = +\infty.$$

We introduce the Heisenberg group in the first section. In the second part of this paper, we complete our point of view on curves of $\mathbb{H}$ and we state two useful lemmas that estimate the distance of points to $\mathbb{H}$-lines. The third part is the construction of the curve $\Omega$ and in the fourth one we use the lemmas of Section 2 for reducing the problem to a planar geometry question and proving Theorem 0.4.

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## 1. Horizontal curves in $\mathbb{H}$

The counterexample of this article is an horizontal curve. In this section we define horizontal curves and give their main properties. By curve we mean the continuous image of a closed interval of $\mathbb{R}$.

1.1. **Definition of $\mathbb{H}$.** We give a naive presentation of $(\mathbb{H}, d_c)$, the (first) Heisenberg group equipped with the Carnot-Carathéodory metric. As a set $\mathbb{H}$ can be written in the form $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ and an element of $\mathbb{H}$ can also be written as $(z,t)$ where $z := x + \mathbf{i}y \in \mathbb{C}$. The group structure of $\mathbb{H}$ is given by

$$(z,t)\cdot(z',t') = (z+z', t+t' - \frac{1}{2}\Im(z\overline{z'}))$$

where $\Im$ denotes the imaginary part of a complex number. $\mathbb{H}$ is then a Lie group with neutral element $0_{\mathbb{H}} := (0,0)$ and inverse element $(-z,-t)$.

Throughout this paper, $\tau_p : \mathbb{H} \to \mathbb{H}$ will be the left translation

$$\tau_p(q) = p \cdot q.$$

For $\lambda > 0$, we denote by $\delta_\lambda$ the dilation

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$$

where $p, q \in \mathbb{H}$ and $\lambda \geq 0$.

We also introduce the rotations

$$\rho_\theta(z,t) = (\mathrm{e}^{\mathbf{i}\theta} z, t)$$

for any $\theta \in \mathbb{R}$.

1.2. **Lifts and projections between $\mathbb{H}$ and $\mathbb{C}$.** We first introduce the complex projection $Z$ from $\mathbb{H}$ to $\mathbb{C}$ defined by

$$Z : (x,y,t) \mapsto (x + \mathbf{i}y).$$

A curve $\gamma(s) = (\gamma_x, \gamma_y, \gamma_t)$ of $\mathbb{H} = \mathbb{R}^3$ is said to be *horizontal* if it is absolutely continuous and

$$\dot{\gamma_t}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{A}\left(Z(\gamma)\right)\right](s)$$

for almost every point $s$. Here $\mathcal{A}\left(Z(\gamma)\right)$ is the algebraic area swept by the curve $\alpha = Z(\gamma)$. It is uniquely defined by $\mathcal{A}(\alpha)(s_0) = 0$ where $s_0$ is the initial time and by the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{A}(\alpha)\right](s) = \frac{1}{2}(\alpha_y \dot{\alpha_x} - \alpha_x \dot{\alpha_y})$$

for almost every $s$.

Similarly we call *planar* curves the absolutely continuous curves of $\mathbb{C}$. The complex projections of horizontal curves $\gamma$ are in particular planar curves. Moreover if one knows $\gamma$ at the initial time and the complex projection $\alpha = Z(\gamma)$, it is possible to recover the whole horizontal curve by the formula giving the third coordinate :

$$\gamma_t(s) = \alpha_t(s_0) + \frac{1}{2}\int_{s_0}^{s} (\alpha_y \dot{\alpha_x} - \alpha_x \dot{\alpha_y}). \tag{2}$$

Then we have the following proposition

**Proposition 1.1.** *Let $p$ be a point of $\mathbb{H}$. We denote by $\Upsilon_p$ the set of horizontal curves $\alpha$ such that $\alpha$ starts in $p$, and $\Upsilon_{p^{\mathbb{C}}}$ the set of planar absolutely continuous curves starting in $p^{\mathbb{C}} = Z(p)$. The projection $Z$ is a bijection from $\Upsilon_p$ to $\Upsilon_{p^{\mathbb{C}}}$.*

We denote by $\mathrm{Lift}_p$ the inverse of $Z$ from $\Upsilon_{p^{\mathbb{C}}}$ to $\Upsilon_p$. We call it the $\mathbb{H}$-*lift* starting from $p$.

1.3. **Direct similitudes.** We introduce the complex direct similitudes

$$\begin{aligned} \delta^{\mathbb{C}}_\lambda(z) &= \lambda z \\ \tau^{\mathbb{C}}_{a+\mathbf{i}b}(z) &= a + \mathbf{i}b + z \\ \rho^{\mathbb{C}}_\theta(z) &= \mathrm{e}^{\mathbf{i}\theta} z. \end{aligned}$$

The complex projection $Z$ almost commutes with $\delta_\lambda$, $\tau_p$ and $\rho_\theta$: we have to replace them by their corresponding complex similitudes. Precisely

$$\begin{aligned} Z(\delta_\lambda(z,t)) &= \delta^{\mathbb{C}}_\lambda(z) \\ Z(\tau_p(z,t)) &= \tau^{\mathbb{C}}_{Z(p)}(z) \\ Z(\rho_\theta(z,t)) &= \rho^{\mathbb{C}}_\theta(z). \end{aligned}$$

As a consequence we have similar relations for $\mathrm{Lift}_p$:

$$\begin{aligned} \delta_\lambda(\mathrm{Lift}_p(\alpha)) &= \mathrm{Lift}_{\delta_\lambda(p)}(\delta^{\mathbb{C}}_\lambda(\alpha)) \\ \tau_p(\mathrm{Lift}_q(\alpha)) &= \mathrm{Lift}_{p\cdot q}(\tau^{\mathbb{C}}_{Z(p)}(\alpha)) \\ \rho_\theta(\mathrm{Lift}_p(\alpha)) &= \mathrm{Lift}_{\rho_\theta}(p)(\rho^{\mathbb{C}}_\theta(\alpha)) \end{aligned}$$

1.4. **Carnot-Carathéodory distance and geodesics.** We define now the metric aspect of $\mathbb{H}$.

**Definition 1.2.** The length of a horizontal curve $\alpha$ of $\mathbb{H}$ is the length in $\mathbb{C}$ of the projected curve $Z(\alpha)$.

As a consequence of Subsection 1.3, the transformations $\delta_\lambda$ multiplies the length of a horizontal curve by $\lambda$. This quantity does not change under the action of $\rho_\theta$ and $\tau_p$.

**Definition 1.3.** The Carnot-Caratheodory distance from $p \in \mathbb{H}$ to $q \in \mathbb{H}$ is the infimum of the length under the horizontal curves going from $p$ to $q$.

Then the Carnot-Carathéodory distance between two points is invariant under the action of $\rho_\theta$ and $\tau_p$. It is multiplied by $\lambda$ if the points are dilated by $\delta_\lambda$.

This infimum in Definition 1.3 is in fact a minimum and the minimizing curve is a $\mathbb{H}$-line or a $\mathbb{H}$-circle as we will see in Proposition 1.4. By $\mathbb{H}$-*line* we mean the

$\mathbb{H}$-lift of a line of $\mathbb{C}$. Similarly a $\mathbb{H}$-*circle* is the $\mathbb{H}$-lift of a circle of $\mathbb{C}$. Here by circles and lines we don't mean the sets but the curves.

**Proposition 1.4.** *For any two points $p$ and $q$ of $\mathbb{H}$, there is a shortest horizontal curve from $p$ to $q$. It is the $\mathbb{H}$-lift of a line or of a circle arc.*

*Proof.* The horizontal curves from $p = (z_p, t_p)$ to $q = (z_q, t_q)$ are exactly the $\mathbb{H}$-lifts starting in $p$ of those absolutely continuous planar curves connecting $Z(p)$ to $Z(q)$ that enclose an algebraic area $t_q - t_p$. Minimizing the length of these curves is the same as minimizing the length in this family of planar curves. This variation problem is strongly related to Dido problem, a very old variant of the planar isoperimetric problem (see Appendix A). □

*Remark* 1.5. We will not prove the following important facts that are widely broadcased. The Carnot-Carathéodory distance is a true distance. It provides the usual topology of $\mathbb{R}^3$. It is a geodesic distance and the length of the horizontal curves is also the length one can define from this distance (the other curves have an infinite length). See [6, 9, 13] for classical presentations introducing the subRiemannian structure.

1.5. **Closed horizontal curves.** If $\alpha$ and $\beta$ are two curves such that the end point of $\alpha$ is the starting point of $\beta$, we define $\alpha\beta$ as the catenation of the two curves. For $\alpha$ defined on $[a, b]$, the reverse curve $\bar{\alpha}$ be defined on $[-b, -a]$ by $\bar{\alpha}(s) = \alpha(-s)$.

**Lemma 1.6.** *Let $z \in \mathbb{C}$, $z' \in \mathbb{C}$ and $(\alpha_1, \alpha_2)$ two planar curves going from $z$ to $z'$, defined respectively on $[a_1, b_1]$ and $[a_2, b_2]$. Then the algebraic area swept by the catenation $\overline{\alpha_2}\alpha_1$ is equal to the third coordinate of*

$$[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)]$$

*for any $\mathbb{H}$-lift* $\mathrm{Lift}(\alpha_1)$ *and* $\mathrm{Lift}(\alpha_2)$ *of $\alpha_1$ and $\alpha_2$ respectively.*

*Proof.* We first assume that both $\mathbb{H}$-lifts $\mathrm{Lift}(\alpha_1)$ and $\mathrm{Lift}(\alpha_2)$ start in a same point $p$ with $Z(p) = z$. Then $\overline{\mathrm{Lift}(\alpha_2)}\,\mathrm{Lift}(\alpha_1)$ is a $\mathbb{H}$-lift of $\overline{\alpha_2}\alpha_1$ and it follows that the planar curve encloses an algebraic area equal to the third coordinate of

$$\begin{aligned}[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [0] =& [\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] \\ & - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)].\end{aligned}$$

The third coordinate difference between two $\mathbb{H}$-lifts of a same planar curve is a constant because of equation (2). The conclusion follows by making a vertical translation of $\mathrm{Lift}(\alpha_1)$ or $\mathrm{Lift}(\alpha_2)$. □

## 2. Geometric Lemmas

In this section we will often use the exponent $^{\mathbb{C}}$ for $Z(\cdot)$. For example, we will write $l^{\mathbb{C}}$ and $q^{\mathbb{C}}$ for the complex projections of $l$ and $q$ respectively.

The orthogonal projection on a line of $\mathbb{C}$ has no obvious horizontally lifted counterpart in $\mathbb{H}$ as we will see now.

**Definition 2.1.** Let $p \in \mathbb{H}$ and $l$ be a $\mathbb{H}$-line. The $\mathbb{C}$-*projection* of $p$ on $l$ is the only point $p^l \in l$ such that $p^{l,\mathbb{C}} := (p^l)^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$.

Now, let $\zeta$ be a planar line. The *lifted-$\mathbb{C}$-projection* of $p$ on $\zeta$ is the only point $p^{\zeta} \in \mathbb{H}$ such that

- $p^{\zeta,\mathbb{C}} := (p^{\zeta})^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on the line $\zeta$
- $p$ and $p^{\zeta}$ are on a $\mathbb{H}$-line

We give an example. The line of equation

$$x = 2 \quad \text{and} \quad t = 3 + y$$

is a $\mathbb{H}$-line. Its complex projection is $x = 2$. The $\mathbb{C}$-projection of the origin $0_{\mathbb{H}} = (0,0,0)$ on this line is $(2,0,3)$. The lifted-$\mathbb{C}$-projection on $x = 2$ is $(2,0,0)$ because $y = t = 0$ is a $\mathbb{H}$-line and its complex projection is orthogonal to $x = 2$.

Notice that like in the previous example, for a given $\mathbb{H}$-line $l$ and a point $p \in \mathbb{H}$, the point $p^{l^{\mathbb{C}}}$ is a well-defined point of $\mathbb{H}$ and that it is not always on $l$. If it is then $p^{l^{\mathbb{C}}} = p^l$ and this point also realizes the distance of $p$ to $l$. In the next lemma, we give pieces of information about the metric projection of a point to a $\mathbb{H}$-line in the general case.

**Lemma 2.2.** *Let $p$ be a point of $\mathbb{H}$ an $l$ a $\mathbb{H}$-line. There is a point $q$ on $l$ that minimizes the distance to $p$. In $q^{\mathbb{C}}$ the $Z$-projection of the unique geodesic between $p$ and $q$ make a right angle with $l^{\mathbb{C}}$.*

*Proof.* It is easier to understand this proof with a look at Figure 2. It represents the situation seen from above, which is equivalent to the planar figure obtained by $Z$-projection. Nevertheless the names of the points and curves are the names of the figure in $\mathbb{H}$. There are many analytical or geometric ways to convince that the distance of $p$ to a point of the $\mathbb{H}$-line tends to $\infty$ at the ends of this line. With a standard compactness argument, we find a point $q$ on $l$ that minimizes the distance to $p$ and let $\gamma$ be the geodesic from $p$ to $q$. We will apply now Lemma 1.6. For the first curve $\alpha_1$, we connect $\alpha := \gamma^{\mathbb{C}}$ with a part of $l^{\mathbb{C}}$ going from $q^{\mathbb{C}}$ to $p^{l,\mathbb{C}} = p^{l^{\mathbb{C}},\mathbb{C}} \in l^{\mathbb{C}}$, the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$. The second curve ($\alpha_2$ in Lemma 1.6) is the segment line from $p^{\mathbb{C}}$ to $p^{l,\mathbb{C}}$. The lemma brings us the following information: our closed curve $\overline{\alpha_2}\alpha_1$ encloses an algebraic area whose value $\mathcal{T}$ is the difference between the third coordinates of $p^l$ and $p^{l^{\mathbb{C}}}$. The Euclidean transposition to our minimizing problem is then equivalent to finding the shortest curve from $p^{\mathbb{C}}$ to $l^{\mathbb{C}}$ such that the algebraic area covered by a moving radius centered in $p^{l,\mathbb{C}}$ is exactly the given quantity $\mathcal{T}$.

The following symmetrization argument using the symmetry with respect to the line $l^{\mathbb{C}}$ and Dido's problem conclude the proof: the shortest symmetric curve from $p^{\mathbb{C}}$ to its symmetric point with respect to $l^{\mathbb{C}}$ that covers the area $2\mathcal{T}$ is a circle arc. The solution is unique if $p^{\mathbb{C}} \notin l^{\mathbb{C}}$ and the curve makes a right angle with $l^{\mathbb{C}}$. $\square$

*Remark* 2.3. Another proof could use the Heisenberg gradient of the distance [2, 14].

We estimate now the distance of a point to a $\mathbb{H}$-line.

**Lemma 2.4.** *Let $p$ be a point of $\mathbb{H}$ and $l$ a $\mathbb{H}$-line. Then the distance of $p$ to the line $l$ is comparable to the Euclidean distance between the projections $p^{\mathbb{C}}$ and $l^{\mathbb{C}}$ plus the distance of the point $p^{l^{\mathbb{C}}}$ obtained by lifted-$\mathbb{C}$-projected to $l$. In fact*

$$\max\left(d_c(p^{\mathbb{C}}, l^{\mathbb{C}}), \frac{d_c(p^{l^{\mathbb{C}}}, l)}{\sqrt{2}}\right) \le d_c(p, l) \le d_c(p^{\mathbb{C}}, l^{\mathbb{C}}) + d_c(p^{l^{\mathbb{C}}}, l).$$

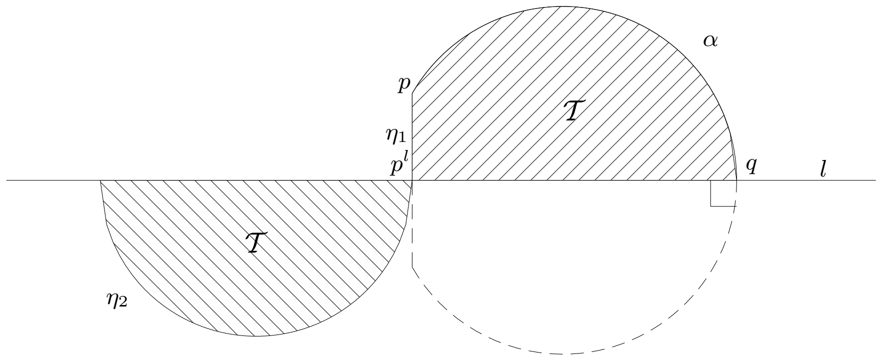


FIGURE 2. Projection lemmas

*Proof.* We use the same notations as in Lemma 2.2. We have in fact to compare the length of $\gamma$ to the sum of the lengths of two curves: $\eta_1$, the $\mathbb{H}$-line segment from $p$ to $p^{l^{\mathbb{C}}}$ and $\eta_2$ one of the two possible shortest curves from $p^{l^{\mathbb{C}}}$ to $l$. The connexion $\eta$ of the two previous curves goes from $p$ to $l$. It follows that the length of $\eta$ is greater than the one of $\gamma$. For the other estimate, we just need to remark than each of the $\eta_i$ is up to a constant smaller than $\gamma$. It is obvious for $\eta_1$ with constant 1. For $\eta_2$ we require one more time Lemma 1.6 and the Dido's problem with a symmetrization in a similar way as in Lemma 2.2. We observe that $\eta_2^{\mathbb{C}}$ describes an half circle and enclose an algebraic area $\mathcal{T}$ as it is represented on Figure 2. We obtain that $\eta_2$ has a length smaller than $\sqrt{2}$ the one of $\alpha$: when we symmetrize $\eta_2^{\mathbb{C}}$ we obtain a circle of area $2\mathcal{T}$. The curve $\alpha^{\mathbb{C}}$ connected with its symmetrization enclose the same area. It minimizes the length if it is an half of circle. The quotient of the lengths of a circle and an half circle with the same area is $\sqrt{2}$. □

We estimate the distance of two points to a $\mathbb{H}$-line.

**Lemma 2.5.** *Let $p_1$ and $p_2$ be two points being on a same $\mathbb{H}$-line and denote another $\mathbb{H}$-line by $l$. Then*

$$d(p_1,l)+d(p_2,l) \geq \frac{d(p_1^{\mathbb{C}},l^{\mathbb{C}})+d(p_2^{\mathbb{C}},l^{\mathbb{C}})+\sqrt{|\mathcal{U}(p_1^{\mathbb{C}}p_1^{l,\mathbb{C}}p_2^{l,\mathbb{C}}p_2^{\mathbb{C}})|}}{2}$$

*where $\mathcal{U}(p_1^{\mathbb{C}}p_1^{l,\mathbb{C}}p_2^{l,\mathbb{C}}p_2^{\mathbb{C}})$ is the algebraic area of the trapezoid $p_1^{\mathbb{C}}p_1^{l,\mathbb{C}}p_2^{l,\mathbb{C}}p_2^{\mathbb{C}}$.*

*Proof.* First of all $d(p_i^{\mathbb{C}},l^{\mathbb{C}}) \leq d(p_i,l)$ for $i \in \{1,2\}$ and we can sum these two relations. It is then enough to prove $d_c(p_1,l)+d_c(p_2,l) \geq \sqrt{|\mathcal{U}(p_1^{\mathbb{C}}p_1^{l,\mathbb{C}}p_2^{l,\mathbb{C}}p_2^{\mathbb{C}})|}$. For that we use Lemma 1.6 where we consider the two following curves (in fact their complex projections): On the one hand the $\mathbb{H}$-line segment of $l$ from $p_1^l$ to $p_2^l$ and on the other hand the $\mathbb{H}$-polygonal line from $p_1^{l^{\mathbb{C}}}$ to $p_2^{l^{\mathbb{C}}}$ going through $p_1$ and $p_2$. Then the algebraic area of the trapezoid is the third coordinate of

$$[p_1^{l^{\mathbb{C}}} - p_1^l] - [p_2^{l^{\mathbb{C}}} - p_2^l]$$

where the sign minus is the difference between two vectors of $\mathbb{R}^3$. Let $\mathcal{T}_i$ be the third coordinate of $[p_i^{l^{\mathbb{C}}} - p_i^l]$ for $i \in \{1, 2\}$ and write simply $\mathcal{U}$ instead of $\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})$. Then there is a $i$ such that $|\mathcal{T}_i| \geq \frac{|\mathcal{U}|}{2}$. For this $i$ we know exactly that the distance of $p_i^{l^{\mathbb{C}}}$ to $l$ is $\sqrt{2\pi|\mathcal{T}_i|}$ (Dido's problem or see the end of Lemma 2.4). Therefore and because of Lemma 2.4, we have $d_c(p_i, l) \geq \frac{d_c(p_i^{l^{\mathbb{C}}}, l)}{\sqrt{2}}$ and finally

$$d_c(p_1, l) + d_c(p_2, l) \geq \frac{1}{\sqrt{2}} \sqrt{2\pi \frac{|\mathcal{U}|}{2}} \geq \sqrt{|\mathcal{U}|}.$$

□

## 3. Construction of $\omega([0, 1])$

As we saw in Section 1, the $\mathbb{H}$-lift provides a direct link between the horizontal curves of $\mathbb{H}$ and the absolutely continuous curves of $\mathbb{C}$. We will describe our curve $\omega$ as the $\mathbb{H}$-lift starting in $\omega(0) = (-1, 0, 0)$ of a planar curve $\omega^{\mathbb{C}}$. This curve is a Von-Koch-like fractal with finite length that we obtain as a limit of certain polygonal lines $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ (see Figure 1 for a representation of $\omega_0^{\mathbb{C}}$, $\omega_1^{\mathbb{C}}$ and $\omega_2^{\mathbb{C}}$). Before we explain the recursive way to build the curves, we precise that $\omega$ and the $\omega_n$ will go from $(-1, 0, 0)$ to $(1, 0, 0)$. The direct consequence is that $\omega^{\mathbb{C}}$ and the $\omega_n^{\mathbb{C}}$ go from $-1$ to $1$ in $\mathbb{C}$.

For the construction of $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$, we require a sequence $(\theta_n)_{n\geq 1}$ of non-negative angles that tends to 0. We start from the simple line segment $\omega_0^{\mathbb{C}} : s \in [0, 1] \mapsto (-1 + 2s, 0, 0)$ and we obtain $\omega_{n+1}^{\mathbb{C}}$ from $\omega_n^{\mathbb{C}}$ in the way we describe below. The curve $\omega_n^{\mathbb{C}}$ is made of $4^n$ segments having the same length. Let us denote this length by $l_n$ and the total length by $L_n = 4^n \cdot l_n$. On the $n + 1$ step we change every segment line by a polygonal line made of 4 segments, having the same beginning and the same end. These four segments have length $\frac{l_n}{4\cos\theta_{n+1}}$ and all make with the former line segment an angle $\theta_{n+1}$ (see Figure 1). There are two ways to respect these conditions. However, the construction is unique if we precise the orientation: when the time grows the first of the 4 small segments make a negative angle with respect to the segment of length $l_n$.

The important remark is that replacing the segment by the polygonal line of 4 segments, we do not change the swept algebraic area.

Let us define the value of the angles $\theta_n$. In all this construction, it will be $\theta_n = \frac{C}{n}$ where $C = 0.2$. We prove now that $\omega^{\mathbb{C}}$ is well-defined as the limit of $\left(\omega_n^{\mathbb{C}}\right)_{n\in\mathbb{N}}$ where each $\omega_n^{\mathbb{C}}$ is parametrized with constant speed on $[0, 1]$.

**Proposition 3.1.** *The sequence of curves $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ tends to a rectifiable curve $\omega^{\mathbb{C}} : [0, 1] \to \mathbb{H}$ parametrized with constant speed.*

*Proof.* The speed of the curves $\omega_n^{\mathbb{C}}$ is exactly the length $L_n$ and this quantity is also the best Lipschitz constant of $\omega^{\mathbb{C}}$. Let us prove the uniform convergence. The curves $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$ meet at every time $\frac{\sigma}{4^n} \in [0, 1]$ where $\sigma = 0, \cdots, 4^n$. Between two subsequent meetings the curve $\omega_{n+1}$ always repeats the same motion pattern while $\omega_n$ is a segment. On $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$ the curves are the more distant when the first of the four segments is done, exactly at time $\frac{\sigma}{4^n} + \frac{1}{4^{n+1}}$. The maximum distance is also attained at time $\frac{\sigma}{4^n} + \frac{3}{4^{n+1}}$. From this observation we deduce

$$\|\omega_n^{\mathbb{C}} - \omega_{n+1}^{\mathbb{C}}\| = (\sin\theta_n) l_{n+1}.$$

The quotient between $l_n$ and $l_{n+1}$ is $\frac{1}{4\cos(\theta_{n+1})}$. Because all $\theta_n$ have a cosine greater than 0.5, this quotient is smaller than $1/2$. We conclude that the series

$$\sum_{n=0}^{+\infty} \|\omega_{n+1}^{\mathbb{C}} - \omega_n^{\mathbb{C}}\| \leq \sum_{n=0}^{+\infty} (\sin\theta_n) l_0 \cdot 2^{-n}$$

converge.

In the next lemma we prove that $L := \limsup_{n\to+\infty} L_n < +\infty$. As a direct consequence $\omega^{\mathbb{C}}$ will be $L$-Lipschitz. We recall that $\theta_n = \frac{C}{n}$ where $C = 0.2$ and with a few trigonometry we see that $L_n = \frac{2}{\prod_{m=1}^n \cos\theta_m}$.

**Lemma 3.2.** *We have $L \leq 2.4 = 1.2 \cdot L_0$. Moreover, $L$ is the optimal Lipschitz constant and the length of $\omega^{\mathbb{C}}$.*

*Proof.* Because of the convexity of log, if $(1-x) \in [\mathrm{e}^{-1}, 1]$, then

$$\log(1-x) \geq \frac{-x}{1-\mathrm{e}^{-1}} \geq -2x.$$

It is possible to apply it to $x = \theta^2/2$ because $\theta \leq C \leq \sqrt{2 - 2\mathrm{e}^{-1}}$. Then we have

$$\log\left(\frac{1}{\prod_{n=1}^N \cos\theta_n}\right) = -\sum_{n=1}^N \log(\cos\theta_n)$$
$$\leq -\sum_{n=1}^N \ln(1 - \frac{\theta_n^2}{2}) \leq \sum_{n=1}^N \theta_n^2 \leq C^2\frac{\pi^2}{6} \leq 0.08.$$

Then we have $L \leq L_0 \exp(0.08) \leq 1.2 \cdot L_0$.

Thus $L$ is the optimal Lipschitz constant for $\omega^{\mathbb{C}}$. Indeed for $m \geq n$ the distance between $\omega^{\mathbb{C}}(\frac{\sigma}{4^n})$ and $\omega^{\mathbb{C}}(\frac{\sigma+1}{4^n})$ is $L_n/4^n$ because

$$\omega^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_m^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_n^{\mathbb{C}}(\frac{\sigma}{4^n}).$$

It follows also from the same observation that $L$ is the length of $\omega^{\mathbb{C}}$. □

□

We defined $\omega$ as the $\mathbb{H}$-lift of $\omega^{\mathbb{C}}$ starting from $(-1,0,0)$ and $\omega_n$ the one of $\omega_n^{\mathbb{C}}$ starting from $(-1,0,0)$. All these curves are parametrized with constant speed on $[0,1]$.

**Lemma 3.3.** *The curves $\omega_n$ and $\omega_{n+1}$ exactly meet on the points $\frac{\sigma}{4^n}$ for $\sigma = 0, \cdots, 4^n$.*

*Proof.* The property is surely true for $\sigma = 0$ because $\omega_{n+1}(0) = \omega_n(0) = (-1,0,0)$. Let $\sigma$ be an integer smaller than $4^n - 1$. We assume that on $[0, \frac{\sigma}{4^n}]$ the curves $\omega_n$ and $\omega_{n+1}$ only meet at the times $\frac{\sigma'}{4^n}$ for $\sigma' = 0, \cdots, \sigma$. Let us now exam what happen on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. The curves are both starting from $\omega_n(\frac{\sigma}{4^n}) = \omega_{n+1}(\frac{\sigma}{4^n})$ and respectively lift $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$. The previous planar curves meet at $\frac{\sigma}{4^n}$, at $\frac{\sigma+1}{4^n}$ and at the mid point $\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}$. Then these are the only possible meeting points for

$\omega_n$ and $\omega_{n+1}$ on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. Now, We consider two $\mathbb{H}$-lifts, starting from $\omega_{n+1}(\frac{\sigma}{4^n})$ and we will use Lemma 1.6 for them. On the one hand we lift horizontally $\omega_{n+1}^{\mathbb{C}}$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}\right]$ and on the other hand we lift $\omega_n^{\mathbb{C}}$ on the same interval. Both planar curves arrive in the same point and the associated closed planar curve sweeps the positive area $(\frac{l_n^2 \cdot \tan(\theta_{n+1})}{4})$ of a triangle. This quantity is the difference for the third coordinate of the end points of the $\mathbb{H}$-lifts. We have

$$\omega_{n+1}(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}) \neq \omega_n(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}).$$

If we make the similar operation lifting $\omega_{n+1}^{\mathbb{C}}$ and $\omega_n^{\mathbb{C}}$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}\right]$, we contrarily obtain an algebraic area equal to zero and can conclude that

$$\omega_{n+1}(\frac{\sigma+1}{4^n}) = \omega_n(\frac{\sigma+1}{4^n}).$$

$\square$

A corollary of this lemma is that for any integer $m \geq n$, $\omega(\frac{\sigma}{4^n}) = \omega_m(\frac{\sigma}{4^n})$.

*Remark* 3.4. In the previous lemma, we noticed that $\omega_{n+1}(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n})$ has the same first coordinates as $\omega_n(\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n})$ but the $t$-coordinate difference is $\frac{l_n^2 \cdot \tan(\theta_{n+1})}{4^2}$. Then the Carnot-Caratheodory distance between them is greater than $\frac{K}{4^n \cdot \sqrt{n}}$ for some constant $K$. It is an indication that the linear segments of $\omega_n$ are not such a good approximation of $\omega$ where a good approximation would have been to be smaller than $\frac{K'}{4^n \cdot n}$ for some constant $K'$. This is a decisive observation and a good reason for believing in Theorem 0.4.

*Remark* 3.5. An amazing observation is that $\omega^{\mathbb{C}}$ is not derivable in any point $\frac{\sigma}{4^n}$ for any $n$ and $\sigma \leq 4^n$. Around these points, the curve is making a spiral because $\sum_{m=n}^{+\infty} \theta_m = +\infty$. However, $\omega^{\mathbb{C}}$ is a Lipschitz curve and is then almost everywhere derivable. In fact it seems that for a time $s \in [0,1]$, written $\overline{0, a_1 a_2 \cdots}^4$ in basis 4, the curve $\omega^{\mathbb{C}}$ is derivable in $s$ if and only if the series $\sum_{m=1}^{+\infty} \frac{\varepsilon(\overline{a_m}^4)}{m}$ converge. Here, $\varepsilon$ is defined by

$$\varepsilon(0) = \varepsilon(3) = 1 \quad \text{and} \quad \varepsilon(1) = \varepsilon(2) = -1.$$

## 4. Counterexample for the inverse implication in [5]

We prove in this section Theorem 0.4, i.e $B_{\mathbb{H}}^{\Delta}(\omega([0,1]))$ is infinite. With the notations of the beginning of this paper, the first step will consist in estimating the cardinal of $\Delta_k$. In the second step, we will estimate from below the value of $\beta_{\mathbb{H}}(x, A \cdot 2^{-k})$ for a $x \in \Delta_k$. For this we will require the geometric lemmas of Section 2.

Because of the second property of the net, $\omega \subset \bigcup_{x \in \Delta_k} \mathcal{B}^{\mathbb{H}}(x, 2^{-k})$. The projection of a ball for the Heisenberg metric on the complex plane is a ball of $\mathbb{R}^2$ with the same radius. That is why

$$\omega^{\mathbb{C}} \subset \bigcup_{x \in \Delta_k} \mathcal{B}^{\mathbb{C}}(x^{\mathbb{C}}, 2^{-k}).$$

If we perform a second projection on the real axis, we obtain that the segment $[-1, 1]$ is covered by a family of segments of length $2^{-k+1}$ which is indexed by $\Delta_k$. We conclude that the cardinal of $\Delta_k$ is greater than $2^k$.

In this paragraph, we examine what is the right fractal scale of the portion of $\omega([0,1])$ intercepted a ball $\mathcal{B}^{\mathbb{H}}(x, A\cdot 2^{-k})$ with center in $\Delta_k$. Let us compare $A\cdot 2^{-k}$ to $\frac{L_\infty}{4^n} \leq \frac{2.4}{4^n}$ and assume $A = 5$ for the rest of this proof. We observe that for every $k > 0$ and $n = \lceil k/2 \rceil$, $\frac{2.4}{4^n}$ is smaller than $A \cdot 2^{-k}$. It follows that there is a $\sigma \in \{0, 1, \cdots, 4^n - 1\}$ such that $\omega\left([\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]\right) \subset \mathcal{B}(x, A\cdot 2^{-k})$.

If we rescale correctly the last portion of curve using the similitudes of the Heisenberg group (Subsection 1.3), we obtain a curve that could have been $\omega$ if we had chosen the sequence of angle $(\theta_{n+m})_{m=1}^{+\infty}$. In particular this curve includes the set $\Lambda_\theta$ made of the five points

$$\left\{(-1;0), (-\frac{1+i\tan(\theta)}{2}; \frac{\tan(\theta)}{2}), (0; \frac{\tan(\theta)}{2}), (\frac{1+i\tan(\theta)}{2}; \frac{\tan(\theta)}{2}), (1;0)\right\}$$

for $\theta = \theta_{n+1}$. We are interested in the maximal distance of one point of $\Lambda_\theta$ to a given $\mathbb{H}$-line $l$. We denote this distance by $d_\theta(l)$ and $D_\theta$ is the minimum of $d_\theta(l)$ over all the $\mathbb{H}$-lines $l$. We noticed that there is a similitude mapping $\Lambda_\theta$ on a part of $\omega \cap \mathcal{B}(x, A\cdot 2^{-k})$. This map multiplies the distances by $\frac{l_n}{2}$ where we recall that $l_n$ is the length of the $4^n$ segments composing $\omega_n$. Then the distance of $\omega \cap \mathcal{B}(x, A\cdot 2^{-k})$ to the closest $\mathbb{H}$-line is greater than $\frac{l_n}{2} D_\theta$ and

$$\begin{aligned} \beta_{\mathbb{H}}(x, A\cdot 2^{-k}) &\geq \frac{l_n}{2}\cdot\frac{D_\theta}{A\cdot 2^{-k}} \\ &\geq \frac{2.4\cdot D_\theta}{4^n\cdot A\cdot 2^{-k}} \\ &\geq \frac{D_\theta}{A}. \end{aligned} \tag{3}$$

**Proposition 4.1.** *Let $\theta < 0.2$ be a positive angle and $l$ a $\mathbb{H}$-line. Then the maximum distance of one of the five points of $\Lambda_\theta$ to $l$ is greater than $K\cdot\sqrt{\theta}$ for some constant $K$ independent of $l$ and $\theta$. In other words*

$$D_\theta \geq K\sqrt{\theta}.$$

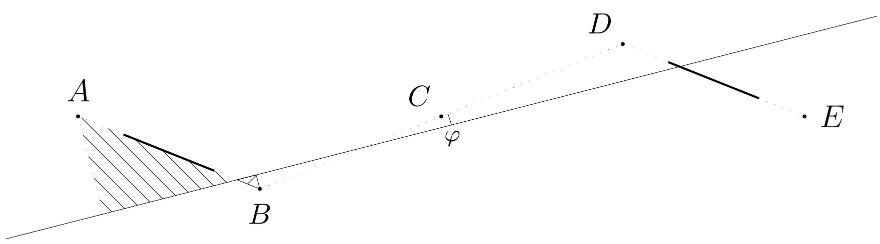


FIGURE 3. The five points are far from a $\mathbb{H}$-line.

*Proof.* In this proof the points of $\mathbb{H}$ will be denoted with capital letters. We will write $A, B, C, D, E$ where we would have wrote $a, b, c, d, e$ before (and $A$ is different from the real constant $A \geq 5$ introduced before). Let us first denote the five points by $A, B, C, D, E$ where $A = (-1, 0, 0)$ and $E = (1, 0, 0)$ like on Figure 3. Thanks to the two geometric lemmas, Lemma 2.4 and Lemma 2.5, we will just have to consider the projections

$$\begin{aligned} A^{\mathbb{C}} &= -1 \\ B^{\mathbb{C}} &= -\frac{1}{2} - i\frac{\tan(\theta)}{2} \\ C^{\mathbb{C}} &= 0 \\ D^{\mathbb{C}} &= \frac{1}{2} + i\frac{\tan(\theta)}{2} \\ E^{\mathbb{C}} &= 1 \end{aligned}$$

and a planar line $l^{\mathbb{C}}$ together with the fact that some points are on a same $\mathbb{H}$-line. It is the case of the couples $(A, B)$, $(D, E)$ and $(A, E)$. The three points $B$, $C$ and $D$ are also on a same $\mathbb{H}$-line.

In this proof, we will sort the possible planar lines $l^{\mathbb{C}}$ by the geometric angle $\varphi \in [0, \frac{\pi}{2}]$ they make with the line $(B^{\mathbb{C}}D^{\mathbb{C}})$. If $\varphi \geq \sqrt{\theta}$, then one of the point $B^{\mathbb{C}}$ or $D^{\mathbb{C}}$ is more distant than $l_\theta \sin\sqrt{\theta}$ to the line $l^{\mathbb{C}}$ where $l_\theta$ is the distance between $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ (it is also the distance between $B$ and $C$ in $\mathbb{H}$ or between $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ for example in $\mathbb{C}$). Then because of Lemma 2.4, the distance of the line $l$ to the farest point is greater than $\frac{1}{2} \cdot (\sqrt{\theta}\frac{2}{\pi})$.

If $\varphi \in [\frac{\theta}{4}, \sqrt{\theta}]$, we consider one of the segment $[B^{\mathbb{C}}C^{\mathbb{C}}]$ or $[C^{\mathbb{C}}D^{\mathbb{C}}]$ that the line $l^{\mathbb{C}}$ does not intersect. Let assume for example, $l^{\mathbb{C}}$ does not intersect $[B^{\mathbb{C}}C^{\mathbb{C}}]$. Then the area of the trapezoid obtained when we project $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ on $l^{\mathbb{C}}$ is greater that $\frac{l_\theta^2 \sin(\varphi)\cdot\cos(\varphi)}{2} \geq \frac{\sin(2\varphi)}{16}$. But $2\varphi \leq 2\sqrt{0.2} \leq \frac{\pi}{2}$. It follows that $\sin(2\varphi) \geq \frac{2\cdot 2\varphi}{\pi}$ and

$$\sqrt{|\mathcal{U}(B^{\mathbb{C}}, B^{\mathbb{C},l}, C^{\mathbb{C},l}, C^{\mathbb{C}})|} \geq \sqrt{\frac{\varphi}{4\pi}} \geq \sqrt{\frac{\theta}{16\pi}},$$

which thanks to Lemma 2.5 provides a lower bound for the distance to $l$ with the right exponent of $\theta$.

The last case, $\varphi \in [0, \frac{\theta}{4}]$ is the more intricate. Here, the line $l^{\mathbb{C}}$ can be very close to $(B^{\mathbb{C}}D^{\mathbb{C}})$. We will prove that it composes a great enough area when projecting orthogonally one of the segments $[A^{\mathbb{C}}B^{\mathbb{C}}]$ or $[D^{\mathbb{C}}E^{\mathbb{C}}]$ on $l^{\mathbb{C}}$. Unlike in the previous case, $l^{\mathbb{C}}$ can intersect both $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[C^{\mathbb{C}}D^{\mathbb{C}}]$. Let assume for a while that $l^{\mathbb{C}}$ can not intersect the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ where we mean by central segment the points on the segment obtained as barycenter of the ends with coefficients between $\frac{1}{4}$ and $\frac{3}{4}$. This assumption is true and we postpone it to Lemma 4.2. Assume for example that $l^{\mathbb{C}}$ does not intercept the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Then projecting $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ on $l^{\mathbb{C}}$, we compose a trapezoid (self-intersecting in the more difficult case as on Figure 3). The angle $\psi$ between $l^{\mathbb{C}}$ and $(A^{\mathbb{C}}B^{\mathbb{C}})$ is included in $[2\theta - \varphi, 2\theta + \varphi]$. This angle $\psi$ is then greater than $\frac{7\theta}{4}$ and smaller than $\frac{\pi}{4}$. Hence we can estimate the algebraic area of the trapezoid in a similar way as in the previous case.

$$\begin{aligned}|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{\mathbb{C},l}A^{\mathbb{C},l})| &\geq \left(\frac{3\cdot l_\theta}{4}\right)^2\frac{\sin(2\psi)}{4}-\left(\frac{l_\theta}{4}\right)^2\frac{\sin(2\psi)}{4}\\ &\geq \frac{\sin(2\psi)}{32}\\ &\geq \frac{2\cdot(2\psi)}{\pi\cdot 32}\geq\frac{7\theta}{32\pi}.\end{aligned}$$

Then we have $\sqrt{|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{\mathbb{C},l}A^{\mathbb{C},l})|}\geq\sqrt{\theta\frac{7}{32\pi}}$ and Lemma 2.5 concludes the proof.

**Lemma 4.2.** *A planar line $l^{\mathbb{C}}$ that makes an angle $\varphi<\frac{\theta}{4}$ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ can not intercept both the central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the one of $[D^{\mathbb{C}}E^{\mathbb{C}}]$.*

*Proof.* We argue by contradiction and assume that $l^{\mathbb{C}}$ intercepts both the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$. We can suppose that $l^{\mathbb{C}}$ goes through $C^{\mathbb{C}}$. Actually as $[A^{\mathbb{C}}B^{\mathbb{C}}]$ is the image of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ by central symmetry, the image $l'^{\mathbb{C}}$ of $l^{\mathbb{C}}$ by the same symmetry has the same property as $l^{\mathbb{C}}$. Namely it goes through the central segments. Moreover, because both central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[D^{\mathbb{C}}E^{\mathbb{C}}]$ are convex, the parallel lines between $l^{\mathbb{C}}$ and $l'^{\mathbb{C}}$ also intercept these two sets. That is why we can assume that $l^{\mathbb{C}}$ is one of the two lines making an angle $\varphi$ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ and going through $C^{\mathbb{C}}$. It's not difficult to convince oneself that $l^{\mathbb{C}}$ can not cross the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Indeed, assume that we divide uniformly $[A^{\mathbb{C}}B^{\mathbb{C}}]$ in four equal parts and join the five points with $C^{\mathbb{C}}$, the greatest of the four angles is the one involving the line $(B^{\mathbb{C}}C^{\mathbb{C}})$. Then it is greater than $\theta/4$ which is the angle average and it is also greater than $\varphi$. This implies a contradiction. □

□

By (3) and Proposition 4.1, we finally get

$$\begin{aligned}B^{\Delta}_{\mathbb{H}}(\omega([0,1])) &\geq \sum_{k\in\mathbb{N}}2^{-k}\sum_{x\in\Delta_k}\beta^2_{\mathbb{H}}(x,A\cdot 2^{-k})(\omega([0,1]))\\ &\geq\sum_{k\in\mathbb{N}}2^{-k}2^k\left(\frac{D_{\theta_{\lceil k/2\rceil+1}}}{A}\right)^2\\ &\geq C\sum_{k\in\mathbb{N}}\frac{1}{\lceil k/2\rceil+1}\geq+\infty.\end{aligned}$$

Hence we have proved Theorem 0.4.

*Remark* 4.3. As we wrote in the introduction there is an analogue of the theory of the geometric salesman problem for metric space, using Menger curvature in the definition of beta numbers [11]. As we did in this paper for the theorem of [5], Schul presented in [16] a counterexample to the converse implication: the criterion of Haholamaa would not be necessary. However this counterexample should not be completely satisfactory (see [16, Subsection 3.3.1]). Anyway it seems that $\Omega$ can not be turned into a counterexample for the approach of Haholamaa.

## Appendix A. Dido problem.

We suppose that we know that the solutions of the planar isoperimetric problem are circles. We consider a very old variant of this problem called Dido's problem [18]. It is related to the foundation of Carthage in Tunisia. It is written that Queen Dido and her followers arrived on a coast by the sea and that the local inhabitants allowed her to stay in as much land as can be encompassed in an oxhide. Then Dido made a rope by cutting the oxhide into fine strips and encircle a wide domain of land. Finding the way to limit this piece of land is a variant of the isoperimetric problem and the optimal way is to make a circle arc. However, the full circle is not optimal because it does not take advantage of the fact that the coast is a natural border. This classical problem of calculus of variation can be reformulate in the following way: consider the curves $\alpha : [0,1] \to \mathbb{C}$ of given length $l$ such that $\alpha(0) = 0_{\mathbb{C}}$ and $\Im(\alpha(1)) = 0$ where $\Im$ is the imaginary part of a complex number. Then the problem is to maximize the algebraic area $\frac{1}{2}\int_0^1 \alpha \times \dot{\alpha}$. Note that in Proposition 1.4 about geodesics of $\mathbb{H}$ we are interested in the dual problem : find the shortest curve enclosing a given area.

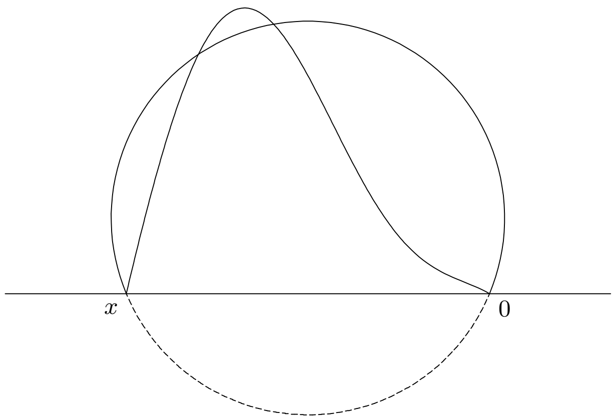


Figure 4. Two curves of same length.

We present here the solutions of Dido's problem and we will see one variant in the next paragraph. The key idea is to close the curve $\alpha$ by connecting it with its symmetric curve with respect to the real line. We obtain a closed curve whose swept area is two times the initial one.

$$\frac{1}{2}\int_0^1 \alpha \times \dot{\alpha} + \frac{1}{2}\int_1^0 \bar{\alpha} \times \dot{\bar{\alpha}} = 2 \cdot \frac{1}{2}\int_0^1 \alpha \times \dot{\alpha}$$

(Here, $\bar{\alpha}$ is the complex conjugated curve. It is not a curve with inverse parametrization defined at the beginning of Subsection 1.5.) The length of this curve is also twice the initial one. If the new curve is a circle, its absolute area is the maximum among all close curves with the same length. This fact is in particular true among the curves symmetric with respect to $y = 0$. It follows that the solution of the authentic Dido's problem is an half of circle. If we now consider the sign of the algebraic area, there are for a given starting point and a given area (positive or negative) exactly two solutions to the problem. These solutions are symmetric with respect to the starting point $0_{\mathbb{C}}$.

In an interesting variant presented on Figure 4, we fix the two ends of the curve. Let us assume for example $\alpha(0) = 0_{\mathbb{C}}$ and $\alpha(1) = x$ for a given $x \in \mathbb{R}^*$. There is an unique circle arc from the first to the second point that encloses a positive algebraic area. The radius of this circle arc is a strictly increasing and continuous function of the length. For a negative area, this is also true. We prove that these circle arcs enclose the greatest possible absolute area. Compare our candidate with another curve and connect both of them with the rest of the circle. Hence we have two closed curves with the same length and one of them is a circle. The area of the circle is greater. Then the circle arc also encloses a greater area as the other curve. We proved that the circle arcs of given length provide the greatest absolute area in this Dido problem with constraint. For positive area as for negative area there is an unique solution curve. In the critical case $x = 0$, the problem is the classical isoperimetric problem. An infinity of circles are solution.

In the dual problem of minimizing the length provided a given area, the solutions are circle arcs as well.

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