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A quadratic version of the traveling salesman problem

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Dedicated to Tudor Zamfirescu on the occasion of his 80th birthday

Abstract

This article presents several results and conjectures in the context of a variant of the traveling salesman problem, in which the cost of travel between two towns is the square of their Euclidean distance.

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1 Introduction

The well-known travelling salesman problem can be formulated as follows. Given an n-tuple of points $(A_1, \ldots, A_n)$ in the Euclidean plane, what is the shortest Hamiltonian tour of it, i.e., for which closed path $[A_{\sigma(1)} \cdots A_{\sigma(n)} A_{\sigma(1)}]$ with some permutation $\sigma \in \mathcal{S}_n$, is the total length

$$|A_{\sigma(1)} A_{\sigma(2)}| + \ldots + |A_{\sigma(n-1)} A_{\sigma(n)}| + |A_{\sigma(n)} A_{\sigma(1)}|$$

minimal? Here EF denotes the line segment between E and F and $|EF|$ its length. The literature about this problem is abundant, as can be seen on the related Wikipedia page.

In the present article we consider a variant of the problem, in which the "cost of travel" between two points E, F is the square $|EF|^2$ of their distance. Given $(A_1, \ldots, A_n)$ we study its *Hamiltonian energy* defined by

$$\mathbf{h}(A_1, \ldots, A_n) = \min_{\sigma \in \mathcal{S}_n} \mathcal{C}\big([A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} A_{\sigma(1)}]\big),$$

where

$$\mathcal{C}\big([B_1 \cdots B_n]\big) = |B_1 B_2|^2 + \cdots + |B_{n-1} B_n|^2 \tag{1}$$

denotes the cost of travel along some polygonal line $[B_1 \cdots B_n]$, closed or not.

Observe that contrary to the classical problem, it makes sense to consider repetitions among the A_i and that adding another point might decrease the Hamiltonian energy. Additionally, for a Hamiltonian tour minimizing the energy, line segments might cross – this is also not the case for the classical problem. These will be detailed in Section 2.

Our main conjecture is the following

Conjecture 1. *For any integer $n > 3$ and any n-tuple $(A_1, \dots, A_n)$ of points in the plane, there exist points A_i, A_j, A_k, not necessarily distinct, among them such that the Hamiltonian energy of $(A_1, \dots, A_n)$ does not exceed the one of (A_i, A_j, A_k), i.e.*

$$\mathbf{h}(A_1, \dots, A_n) \leqslant \mathbf{h}(A_i, A_j, A_k).$$

In fact we wish to formulate a slightly stronger conjecture. First, observe that there is a unique smallest disk $\mathcal{D}$ containing our n-tuple, and that the circle bounding $\mathcal{D}$ contains either two points of the n-tuple forming a diameter, or (at least) three points containing the center of $\mathcal{D}$ in their convex hull, or both. Then our second conjecture is as follows.

Conjecture 2. *Given an integer $n > 3$ and an n-tuple $(A_1, \dots, A_n)$, let $\mathcal{D}$ be the smallest disk containing $\{A_1, \dots, A_n\}$.*

If A_1, A_2, A_3 are on the boundary of $\mathcal{D}$ and if the center of $\mathcal{D}$ is in their convex hull then $\mathbf{h}(A_1, \dots, A_n) \leqslant \mathbf{h}(A_1, A_2, A_3)$.

If A_1A_2 is a diameter of $\mathcal{D}$, then $\mathbf{h}(A_1, \dots, A_n) \leqslant \mathbf{h}(A_1, A_2)$.

It is easy to prove Conjecture 2 when the n points are in convex position, see Proposition 1. We will also prove Conjecture 2 when three or four of the points contain all the points in their convex hull, see Corollary 10. The simplest unclear case is the one of five points in convex position containing a sixth point in their convex hull. We will prove Conjecture 2 in this case under the additional assumption that two of the points form a diameter of $\mathcal{D}$, see Corollary 16.

We reformulate Conjecture 1 using the following notation. Given a nonempty bounded subset $\mathcal{K}$ of the plane and an integer $n \in \mathcal{N}$, $n \geqslant 2$, let us introduce

$$\mathbf{H}_n(\mathcal{K}) = \sup \big\{ \mathbf{h}(A_1, \dots, A_n) \ ; \ A_i \in \mathcal{K} \big\}. \tag{2}$$

By definition each function $\mathbf{H}_n$, $n \geqslant 2$, is nondecreasing for the inclusion: If $\mathcal{K} \subseteq \mathcal{L}$ then $\mathbf{H}_n(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{L})$. Nevertheless, we were able to prove the monotonicity of the sequence $\big(\mathbf{H}_n(\mathcal{K})\big)_{n \in \mathcal{N}}$ only for the few first terms, see Section 2 for details. For this reason, we put

$$\mathbf{H}(\mathcal{K}) = \sup_{n \in \mathcal{N}} \mathbf{H}_n(\mathcal{K}). \tag{3}$$

If $\mathcal{K}$ is a rectangle, it turns out that the whole sequence $\big(\mathbf{H}_n(\mathcal{K})\big)_{n \geqslant 2}$ is constant, see Corollary 9.

Conjecture 1 now is straightforwardly equivalent to

Conjecture 3. *(Conjecture 1 reformulated). For any nonempty bounded subset $\mathcal{K}$ of the plane, we have*

$$\mathbf{H}(\mathcal{K}) = \mathbf{H}_3(\mathcal{K}).$$

A weaker conjecture is the following.

Conjecture 4. *For any nonempty bounded subset $\mathcal{K}$ of the plane, there exists an integer $N = N(\mathcal{K})$ such that* $\mathbf{H}(\mathcal{K}) = \mathbf{H}_N(\mathcal{K})$.

The notion of Hamiltonian energy is closely related to the one of *chain energy*. Let us fix two points A, B of the plane, let $n \in \mathcal{N}$, and let $(A_1, \ldots, A_n)$ be an n-tuple of points of the plane. The *chain energy of* $(A_1, \ldots, A_n)$ *associated with* AB, denoted by $\mathbf{c}_{AB}(A_1, \ldots, A_n)$, is the minimal cost of a Hamiltonian chain starting from A, passing once through all the A_i, and ending at B:

$$\mathbf{c}_{AB}(A_1, \ldots, A_n) = \min_{\sigma \in \mathcal{S}_n} \mathcal{C}\big([AA_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(n)}B]\big). \tag{4}$$

Given an integer $n \geqslant 1$, we say that a bounded subset $\mathcal{K}$ of the plane *satisfies Property* $\mathbf{P}_n$ if, for every n-tuple $(A_1, \ldots, A_n) \in \mathcal{K}$, one has

$$\mathbf{c}_{AB}(A_1, \ldots, A_n) \leqslant |AB|^2.$$

We say that $\mathcal{K}$ *has Property* $\mathbf{P}$ if it satisfies $\mathbf{P}_n$ for all $n \in \mathcal{N}$.

Since Property $\mathbf{P}_n$ is stable by inclusion for each fixed n (i.e., if $\mathcal{K}$ satisfies $\mathbf{P}_n$ then any subset of $\mathcal{K}$ does), it may be of interest to study subsets $\mathcal{K}$ satisfying $\mathbf{P}_n$ that are maximal for the inclusion.

As is the case for $\mathbf{H}_n$, except for the few first values of n, it is unclear if $\mathbf{P}_n$ implies $\mathbf{P}_k$ for all $k \leqslant n$.

Several results and conjectures on the properties $\mathbf{P}_n$ are presented in Sections 3 and 4. Here we only mention a few.

The fact that right triangles of hypothenuse AB have Property $\mathbf{P}$ is mentioned e.g. in [3] and in [4] (exercise 57). As we were not able to find a work in the literature where it is thoroughly proved, we present such a proof below Theorem 8 and analyze the cases of equality. The proof follows the same algorithmic steps as the construction of Pólya's space-filling curve [5]. According to [1, 2] this observation and the result — probably for the right isosceles triangles only — date back at least to an unpublished work of Kakutani in 1966. Theorem 8 implies Conjecture 3 for rectangles, see Corollary 9. In [3] Kahane asked for the value of $\mathbf{H}(\mathcal{K})$ for other sets, which lead us to formulate Conjecture 3. Theorem 12 shows that 1/2-Hölder space-filling curves cannot help proving Property $\mathbf{P}$ for larger sets than right triangles. Therefore it establishes a serious limit for the until now unique available approach.

We refer to [6, 7] for a worst case analysis in higher dimensions and when the travel cost is the p-power of the distance, $p \neq 2$.

One of our main results is that a half-disk of diameter AB has Property $\mathbf{P}_3$, see Theorem 15. Its proof is surprisingly difficult and given in the appendix.

2 Hamiltonian energy

We begin this section by two useful formulae. If A, B, M are three points of the plane, then one finds

$$\mathcal{C}\big([AMB]\big) - \mathcal{C}\big([AB]\big) = -2\overrightarrow{AM}.\overrightarrow{MB}. \tag{5}$$

This cost difference is negative if and only if $|\widehat{AMB}| \geqslant \pi/2$, i.e., if and only if M is in the closed disk of diameter AB. As already said, this means that the cost of a path can be reduced by adding new points. The second formula is the following: If I is the midpoint of A and B, then we have

$$\mathcal{C}\big([AMB]\big) = 2|MI|^2 + \tfrac{1}{2}|AB|^2. \tag{6}$$

As a consequence, the level lines of the function $M \mapsto \mathcal{C}\big([AMB]\big)$ are circles centered at I.

A consequence of (5) is that, for $n \geqslant 3$, the Hamiltonian energy of an n-tuple can be decreased just by repeating some points: Obviously we have $\mathbf{h}(A,B) = \mathbf{h}(A,A,B)$ for any two points A, B, but, already for three points A, B, C we have:

$$\text{If } \ |\widehat{BAC}| > \frac{\pi}{2} \ \text{ then } \ \mathbf{h}(A,B,C) > \mathbf{h}(A,A,B,C). \tag{7}$$

Of course the repetition of points never increases the Hamiltonian energy. In other words, we have for all n-tuples $(A_1, \ldots, A_n)$:

$$\text{If } \ m < n \ \text{ and } \ \{A_1, \ldots, A_m\} = \{A_1, \ldots, A_n\} \ \text{ then } \ \mathbf{h}(A_1, \ldots, A_n) \leqslant \mathbf{h}(A_1, \ldots, A_m). \tag{8}$$

In this context, it is useful to introduce the *support* $\{A_1, \ldots, A_n\}$ of an n-tuple $(A_1, \ldots, A_n)$.

Another fact is that the Hamiltonian energy of an n-tuple may be reached by Hamiltonian tours with crossing edges: For instance, if $\varepsilon > 0$ is small enough and A, B, C, D are the points of coordinates $(0,0)$, $(1,\varepsilon)$, $(2,\varepsilon)$, and $(3,0)$, respectively, then $\mathbf{h}(A,B,C,D)$ is close to 10, reached by the closed path $[ABDCA]$ whose segments AC and BD cross, while the noncrossing path $[ABCDA]$ has a cost close to 12.

Proposition 1. *Conjecture 2 is true if the n points are in convex position.*

Proof. Let $[A_1 \cdots A_n]$ be a convex n-gon, ordered counterclockwise, and let $\mathcal{D}$ be the smallest disk containing it. We have $\mathbf{h}(A_1, \ldots, A_n) \leqslant \mathcal{C}\big([A_1 \cdots A_n]\big)$.

We treat the case where the boundary of $\mathcal{D}$ contains at least three distinct points $A = A_i$, $B = A_j$ and $C = A_k$, $1 \leqslant i < j < k \leqslant n$. If the support $\{A_1, \ldots, A_n\}$ is reduced to $\{A, B, C\}$ we are done. Otherwise let A_ℓ be a distinct point with, say, $i < \ell < j$. Since the angle $\widehat{BA_\ell A}$ is at least $\pi/2$ and the points are in convex position, we have $\widehat{A_{\ell+1}A_\ell A_{\ell-1}} \geqslant \frac{\pi}{2}$ and formula (5) shows that

$$\mathcal{C}\big([A_1 \cdots A_{\ell-1}A_\ell A_{\ell+1} \cdots A_n]\big) \leqslant \mathcal{C}\big([A_1 \cdots A_{\ell-1}A_{\ell+1} \cdots A_n]\big).$$

This can be repeated until only the three points A, B, C remain, for which we have $\mathcal{C}\big([ABC]\big) = \mathbf{h}(A,B,C)$.

The case of two points on the boundary of $\mathcal{D}$ is similar. $\square$

The lemma below follows directly from (5) and the definition of $\mathbf{h}$.

Lemma 2. *Let $(A_1, \ldots, A_n)$ be an n-tuple of the plane and M another point*[1] *such that $|\widehat{A_iMA_j}| \leqslant \pi/2$ for every $i, j \in \{1, \ldots, n\}$ such that $A_i \neq M \neq A_j$. Then we have*

$$\mathbf{h}(A_1, \ldots, A_n, M) \geqslant \mathbf{h}(A_1, \ldots, A_n).$$

[1] not necessarily outside the support $\{A_1, \ldots, A_n\}$ of $(A_1, \ldots, A_n)$

Observe that the statement allows M to be one of the A_i; in that case we obtain equality. This results yields a short proof of the following inequalities, valid for any bounded subset $\mathcal{K}$ of the plane:

$$\forall n \geqslant 5 \qquad \mathbf{H}_2(\mathcal{K}) \leqslant \mathbf{H}_3(\mathcal{K}) \leqslant \mathbf{H}_4(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{K}). \tag{9}$$

Indeed, for all $A, B \in \mathcal{K}$ we have $\mathbf{h}(A,B) = \mathbf{h}(A,A,B) \leqslant \mathbf{H}_3(\mathcal{K})$, yielding the first inequality. Given an arbitrary (possibly degenerate) triangle $A, B, C \in \mathcal{K}$, one of its unsigned angles, say $|\widehat{CAB}|$, is less than $\pi/2$ and we have

$$\mathbf{h}(A,B,C) = \mathbf{h}(\overbrace{A,\dots,A,}^{n-2 \text{ times}}\ B,C) \leqslant \mathbf{H}_n(\mathcal{K}),$$

showing that $\mathbf{H}_3(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{K})$ for any bounded $\mathcal{K}$. Similarly, given $A, B, C, D \in \mathcal{K}$ arbitrary, if one, say D, is in the convex hull of the others, then we already have

$$\mathbf{h}(A,B,C,D) \leqslant \mathbf{h}(A,B,C) \leqslant \mathbf{H}_3(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{K}),$$

and if they are in convex position and ordered, then one of the unsigned angles, say $|\widehat{DAB}|$, is at most $\pi/2$ and we have

$$\mathbf{h}(A,B,C,D) = \mathbf{h}(\overbrace{A,\dots,A}^{n-3 \text{ times}}, B,C,D) \leqslant \mathbf{H}_n(\mathcal{K})$$

for any $n \geqslant 5$, proving the last inequality. We can go one step further:

Proposition 3. *For any bounded $\mathcal{K}$ and any $n \geqslant 5$, one has $\mathbf{H}_5(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{K})$.*

Proof. Given $A, B, C, D, E \in \mathcal{K}$ arbitrary, if one of the points, say A, satisfies the same property as M in Lemma 2, then we have

$$\mathbf{h}(A,B,C,D,E) = \mathbf{h}(A,\dots,A,B,C,D,E) \leqslant \mathbf{H}_n(\mathcal{K})$$

as before. Otherwise the five points form a convex pentagon of vertices labelled in the order A, B, C, D, E, and of angles on the boundary $|\widehat{EAB}|, \dots, |\widehat{DEA}|$ all five greater than $\pi/2$. Without loss of generality, we assume that the diagonal $|EB|$ is (one of) the smallest among the five diagonals. Then one checks that both costs $\mathcal{C}\big([BCEDB]\big)$ and $\mathcal{C}\big([BDCEB]\big)$ are $\geqslant \mathcal{C}\big([BCDEB]\big)$, hence $\mathbf{h}(B,C,D,E)$ is reached by the non-crossing path $[BCDEB]$, and we obtain

$$\mathbf{h}(A,B,C,D,E) \leqslant \mathcal{C}\big([BCDEAB]\big) < \mathcal{C}\big([BCDEB]\big) = \mathbf{h}(B,C,D,E) \leqslant \mathbf{H}_4(\mathcal{K}) \leqslant \mathbf{H}_n(\mathcal{K})$$

by (9). $\square$

We have not pursued our investigations any further; a fortiori we do not know whether the sequence $\big(\mathbf{H}_n(\mathcal{K})\big)_{n\in\mathcal{N}}$ is nondecreasing or not for a general $\mathcal{K}$.

This kind of monotonicity becomes natural if we slightly modify our definition of Hamiltonian energy as follows. A *covering tour* on an n-tuple $(A_1, \dots, A_n)$ is a closed path passing through each vertex A_i at least once, i.e., a closed path with $N \geqslant n$ vertices of the form $[A_{\varphi(1)} \cdots A_{\varphi(N)} A_{\varphi(1)}]$, where φ belongs to $\mathbf{SM}(N,n)$, the set of surjective maps from

$\{1, \dots, N\}$ onto $\{1, \dots, n\}$. The *covering energy* of $(A_1, \dots, A_n)$, denoted by $\mathbf{s}(A_1, \dots, A_n)$, is the infimum of the costs of all covering tours:

$$\mathbf{s}(A_1, \dots, A_n) = \inf_{N \geqslant n} \min_{\varphi \in \mathbf{SM}(N,n)} \mathcal{C}\big([A_{\varphi(1)} \cdots A_{\varphi(N)} A_{\varphi(1)}]\big). \tag{10}$$

Since Hamiltonian tours are particular covering tours, we obviously have $\mathbf{s} \leqslant \mathbf{h}$. We also clearly have that, for a fixed bounded set $\mathcal{K}$, the sequence $(\mathbf{S}_n(\mathcal{K}))_{n \in \mathcal{N}}$ given by

$$\mathbf{S}_n(\mathcal{K}) = \sup \big\{\mathbf{s}(A_1, \dots, A_n) \ ; \ A_i \in \mathcal{K}\big\}$$

is nondecreasing. A consequence of the following result is that the infimum in (10) is actually a minimum, reached for some $N \leqslant 3n$.

Theorem 4. *Given a covering tour on an n-tuple $(A_1, \dots, A_n)$, there is a covering tour on the same n-tuple and of same or lower cost which passes through the points at most three times.*

For a proof, we repeatedly erase redundancies using the following lemma.

Lemma 5. *Let $(A_1, \dots, A_n)$ be an n-tuple and $(B, \dots, B)$ an m-tuple of coinciding points, with $m \geq 4$. Then we have*

$$\mathbf{h}(A_1, \dots, A_n, B, B, B) = \mathbf{h}(A_1, \dots, A_n, \overbrace{B, \dots, B}^{m \ times}). \tag{11}$$

Proof. By (8) we only have to prove that the left hand side is less than or equal to the right hand side. Consider a minimal Hamiltonian tour for the $(n+m)$-tuple $(A_1, \dots, A_n, B, \dots, B)$. Without loss of generality we can assume that B is none of the A_i and that the tour starts (and ends) at B and that the labels of the A_i increase. Hence the tour is of the form

$$\big[B^{m_1} A_1 \cdots A_{n_1} B^{m_2} A_{n_1+1} \cdots A_{n_2} B^{m_3} \cdots B^{m_k} A_{n_{k-1}+1} \cdots A_{n_k} B^{m_{k+1}}\big]$$

with $m_1 + \dots + m_{k+1} = m + 1$ and $n_k = n$. In this labelling, k is the number of visits of the tour at B and the exponents on B indicate the number of times the tour stays on B at each visit before leaving it. Of course this tour has the same cost as the tour

$$\big[BA_1 \cdots A_{n_1} BA_{n_1+1} \cdots A_{n_2} B \cdots BA_{n_{k-1}+1} \cdots A_{n_k} B\big] \tag{12}$$

If $k \leqslant 3$, we are done. Otherwise, we want to find a Hamiltonian tour for the $(m + k - 1)$-tuple $(A_1, \dots, A_n, B, \dots, B)$, i.e. which skips B once, which contains all the others vertices, possibly in a different order, and which is of cost less than or equal to the cost of the tour given by (12). This tour (12) is made of k so-called *subtours* $[BA_{n_i+1} \cdots A_{n_{i+1}} B]$. Observe that these subtours can be made in any order and that each one can also be made in the reversed order without modifying the cost of the tour.

The idea is to modify the tour (12) by merging two subtours and skipping the visit at B between them. This can be done if among the $2k$ ends of subtours $A_1, A_{n_1}, A_{n_1+1}, \dots, A_{n_k}$ we find two points A_i and A_j not belonging to the same subtour and such that the angle $|\widehat{A_i B A_j}|$ is at most $\pi/2$. Then by removing the edges A_iB, A_jB and adding the edge A_iA_j we obtain the desired tour.

For this purpose, we group these $2k$ points in pairs

$$(A_1, A_{n_1}), (A_{n_1+1}, A_{n_2}), \dots, (A_{n_{k-1}+1}, A_{n_k}),$$

each pair being the pair of ends of a chain not containing B. Some of these points may coincide, but they are different from B by assumption. We now label these $2k$ points differently: by the (signed) angle they do with B, i.e. we rename them $A^{\theta_1}, \dots, A^{\theta_{2k}}$ with $0 = \theta_1 \leq \cdots \leq \theta_{2k} \leq 2\pi$, where $\theta_i = \widehat{A^{\theta_1} B A^{\theta_i}} \in [0, 2\pi[$. We have $\sum_{i=1}^{2k} (\theta_{i+2} - \theta_i) = 4\pi$ where we adopt the periodic notation $\theta_{2k+1} = \theta_1$ and $\theta_{2k+2} = \theta_2$. If $k \geqslant 4$, then there exists at least one value of i for which $\theta_{i+2} - \theta_i \leq \pi/2$. It follows that, among the three points $A^{\theta_i}, A^{\theta_{i+1}}, A^{\theta_{i+2}}$, at least two of them denoted by A_i and A_j, make an unsigned angle $|\widehat{A_i B A_j}| \leqslant \pi/2$ and do not belong to the same pair. □

3 Chain energy

In this section we fix two distinct points A, B in the plane. We recall the chain energy $\mathbf{c}_{AB}(A_1, \dots, A_n)$ associated with AB of an n-tuple $(A_1, \dots, A_n)$, given by (4). Analogously to formula (8) we have for all n-tuples $(A_1, \dots, A_n)$:

$$\text{If } \ m < n \ \text{ and } \ \{A_1, \dots, A_m\} = \{A_1, \dots, A_n\} \ \text{ then } \ \mathbf{c}_{AB}(A_1, \dots, A_n) \leqslant \mathbf{c}_{AB}(A_1, \dots, A_m) \tag{13}$$

and similarly to Lemma 2 we have

Lemma 6. *Let $(A_1, \dots, A_n)$ be an n-tuple of the plane and M a distinct further point such that $|\widehat{NMP}| \leqslant \pi/2$ for every $N, P \in \{A_1, \dots, A_n, A, B\}$. Then we have*

$$\mathbf{c}_{AB}(A_1, \dots, A_n, M) \geqslant \mathbf{c}_{AB}(A_1, \dots, A_n).$$

Recall also that a bounded set $\mathcal{K}$ has Property $\mathbf{P}_n$ if for every n-tuple $(A_1, \dots, A_n) \in \mathcal{K}$

$$\mathbf{c}_{AB}(A_1, \dots, A_n) \leqslant |AB|^2,$$

and that $\mathcal{K}$ has Property $\mathbf{P}$ if it has Property $\mathbf{P}_n$ for all $n \in \mathcal{N}$.

We are mainly interested in Property $\mathbf{P}$, but already the properties $\mathbf{P}_n$ for some small values of n are of interest.

Remarks.

1. The disk $\mathcal{D}_{AB}$ of diameter AB has Property $\mathbf{P}_1$ but not $\mathbf{P}_2$. It is even maximal with Property $\mathbf{P}_1$ in the following sense: If a compact $\mathcal{K}$ satisfies $\mathbf{P}_1$ (i.e. $\mathbf{c}_{AB}(C) \leqslant |AB|^2$ for every $C \in \mathcal{K}$) then $\mathcal{K}$ is included in $\mathcal{D}_{AB}$. On the contrary, if CD is a different diameter of the same disk, then $\mathbf{c}_{AB}(C, D) > |AB|^2$.

2. It follows that, if $\mathcal{K}$ has Property $\mathbf{P}_2$ and contains A and B, then AB is the unique diameter of $\mathcal{K}$.

It is easy to construct bounded sets that have Property $\mathbf{P}_2$ but not $\mathbf{P}_3$. It is less easy to find *convex* sets with this property.

Theorem 7. *With $A = (-1, 0)$ and $B = (1, 0)$, the filled ellipse $\mathcal{E} = \{(x, y) \in \mathbb{R}^2 ; x^2 + 3y^2 \leqslant 1\}$ satisfies $\mathbf{P}_2$ but not $\mathbf{P}_3$.*

This ellipse satisfies also the following maximality property: If a bounded set $\mathcal{K}$ symmetric with respect to AB satisfies Property $\mathbf{P}_2$, then $\mathcal{K} \subseteq \mathcal{E}$.

Proof. $\mathcal{E}$ is maximal. If $\mathcal{K}$, symmetric with respect to AB, satisfies $\mathbf{P}_2$ and $M = (x, y) \in \mathcal{K}$, then $N = (x, -y) \in \mathcal{K}$ and, since $AB \perp MN$, we have

$$\mathbf{c}_{AB}(C, D) = \mathcal{C}\big([AMNB]\big) = \mathcal{C}\big([ANMB]\big).$$

The calculation gives $\mathbf{c}_{AB}(C, D) = 2 + 2x^2 + 6y^2$. Then $\mathbf{P}_2$ for $\mathcal{K}$ implies that $M \in \mathcal{E}$.

$\mathcal{E}$ satisfies $\mathbf{P}_2$. Consider two points $C, D \in \mathcal{E}$. Without loss of generality, one can assume that C is on the upper part of the boundary of $\mathcal{E}$, D on its lower part, and that their projections C' and D' on AB are in the order $AC'D'B$. In this manner we have $\mathbf{c}_{AB}(C, D) = \mathcal{C}\big([ACDB]\big) \leqslant \mathcal{C}\big([ADCB]\big)$. With $f(x) = \sqrt{\frac{1-x^2}{3}}$, these points are $C = (x, f(x))$ and $D = (y, -f(y))$, with $-1 \leqslant x \leqslant y \leqslant 1$, and we have to prove that $|AC|^2+|CE|^2+|EB|^2 \leqslant 4$. After simplification, this amounts to showing that

$$r(x, y) := 1 - 2x^2 - 3x - 2y^2 + 3y + 3xy \geqslant f(x)f(y).$$

Writing $r(x, y) = 1 - x^2 + (y - x)(3 + x - 2y)$ shows that $r(x, y) \geqslant 0$ and it remains to prove that $r(x, y)^2 \geqslant f(x)^2 f(y)^2 = (1 - x^2)(1 - y^2)$. Neglecting the term $\big((y - x)(3 + x - 2y)\big)^2$, we are done if we prove that

$$(1 - x^2)^2 + 2(1 - x^2)(y - x)(3 + x - 2y) \geqslant (1 - x^2)(1 - y^2),$$

i.e., after factorization, that $(1 - x^2)(y - x)(6 + 3x - 3y) \geqslant 0$.

$\mathcal{E}$ does not satisfy $\mathbf{P}_3$. Choose $x = \frac{1}{4}(\sqrt{7} - 1)$, $y = f(x) = \frac{1}{4\sqrt{3}}(\sqrt{7} + 1)$ and consider the following points in $\mathcal{E}$: $C = (-x, y)$, $D = (0, -1/\sqrt{3})$ and $E = (x, y)$. Together with A and B, they form an $\mathcal{M}$ in the sense of Lemma 14, that is $AE \perp CD$ and $DE \perp CB$. As shown in the remark following it, this implies that

$$\mathbf{c}_{AB}(C, D, E) = \mathcal{C}\big([ADCEB]\big) = \mathcal{C}\big([ACDEB]\big) = \mathcal{C}\big([ACEDB]\big).$$

The calculation then shows that $\mathbf{c}_{AB}(C, D, E) = \frac{20}{3} - \sqrt{7} \approx 4.02 > 4$. □

The following result is well known. Classically, its proof relies on properties of Pólya's space filling curves. For self-containedness we provide an elementary proof.

Theorem 8. *A right triangle of hypothenuse AB has Property $\mathbf{P}$.*

More precisely, if the points $A_1, \ldots, A_n$ belong to the right triangle $\Delta = ABC$ of hypothenuse AB, then $\mathbf{c}_{AB}(A_1, \ldots, A_n) \leqslant |AB|^2$, with equality if and only if:

$$\textit{either} \quad \{A_1, \ldots, A_n\} \subseteq \{A, B, C\} \quad \textit{or} \quad \{A, B, A_1, \ldots, A_n\} = \{A, B, C, H\}, \tag{14}$$

where H is the orthogonal projection of C on AB.

Proof. We assume that all the A_i are distinct, see (13). Our proof relies on a dyadic partition of Δ. We call Δ_0 the (closed) triangle AHC and Δ_1 is CHB. Note that both triangles are similar to ABC, that $\Delta = \Delta_0 \cup \Delta_1$, and that $\Delta_0 \cap \Delta_1 = CH$. This operation can be iterated so that one gets 2^k similar triangles after k iterations. Since the diameter of the largest triangle of generation k tends to zero as k tends to infinity, we can use this procedure to separate any finite family $\mathcal{F} \subset \Delta$ of distinct points. For k large enough, there is an injective map from $\mathcal{F}$ to the set of triangles of generation k — possibly not unique since some points may lie on border segments like CH.

We prove the result by induction. Let (P_k) be the following property:

For every right triangle $\Delta = PQR$ with hypothenuse PQ and any finite subset $\mathcal{E} = \{A_1, \dots, A_n\}$ of Δ such that $\mathcal{E}^* := \mathcal{E} \setminus \{P, Q\}$ can be separated by the triangles of generation k, we have $\mathbf{c}_{PQ}(A_1, \dots, A_n) \leqslant |PQ|^2$, and equality occurs only in one of the situations analogous to (14).

Note that the statement to prove corresponds to (P_k) for every $k \in \mathcal{N}$.

Property (P_0): There is one triangle, namely $\Delta = PQR$, so that $\mathcal{E}^*$ must have cardinality 0 or 1. In the less trivial case, the equality comes from (5) and the case of equality occurs if and only if the point is R. This proves (P_0).

$(P_k) \Rightarrow (P_{k+1})$: Fix $k \in \mathcal{N}$, assume Property (P_k). Let us prove (P_{k+1}) for the triangle $\Delta = ABC$. Let $\mathcal{E} = \{A_1, \dots, A_n\}$ be a finite set of points in Δ such that the points of $\mathcal{E}^*$ can be injectively mapped to the 2^{k+1} triangles of generation $k+1$. We separate $\mathcal{E}^*$ in two disjoint subsets $\mathcal{E}_0^* \subseteq \Delta_0$ and $\mathcal{E}_1^* \subseteq \Delta_1$ (if A_i is on CH, we put it indifferently in $\mathcal{E}_0$ or in $\mathcal{E}_1$). Applying (P_k) for the triangles ACH and CBH, respectively, we obtain

$$\mathbf{c}_{AC}(\mathcal{E}_0^*) \leqslant |AC|^2 \quad \text{and} \quad \mathbf{c}_{CB}(\mathcal{E}_1^*) \leqslant |CB|^2.$$

Therefore there exists a chain from A to B going through $C, A_1, \dots, A_n$ with chain energy at most $|AC|^2 + |CB|^2 = |AB|^2$. By Lemma 6 we now erase point C from the chain.

It remains to prove the cases of equality. This can only happen when the equality cases occur both in Δ_0 and Δ_1. Therefore, using (P_k) we obtain a finite number of configurations where equality may occur, with points belonging to $\{A, B, C, H, H_0, H_1\}$, where H_0 and H_1 are the orthogonal projections of H on AC and CB, respectively. Now a case-by-case analysis shows that only the cases listed in the statement yield equality. □

Corollary 9. *Let $\mathcal{R}$ be a rectangle in the plane.*

a. *We have $\mathbf{H}_n(\mathcal{R}) = \mathbf{H}_2(\mathcal{R})$ for all $n > 2$.*

b. *The n-tuples that realize $\mathbf{H}_2(\mathcal{R})$ have a support of cardinality at most 5. More precisely, an n-tuple $(A_1, \dots, A_n) \in \mathcal{R}^n$ satisfies $\mathbf{h}(A_1, \dots, A_n) = \mathbf{H}_2(\mathcal{R})$ if and only if its support $\{A_1, \dots A_n\}$ is of cardinality at most 5 and, either is made of four, three or two opposite vertices of $\mathcal{R}$, or $\mathcal{R}$ is a square and this support is made of all the four vertices plus the center of $\mathcal{R}$.*

Proof. **a.** Let A, B, C, D denote the vertices of $\mathcal{R}$, ordered counterclockwise, and consider an n-tuple $(A_1, \dots, A_n)$ of points of $\mathcal{R}$. By Lemma 2 we have

$$\mathbf{h}(A_1, \dots, A_n) \leqslant \mathbf{h}(A_1, \dots, A_n, A) \leqslant \mathbf{h}(A_1, \dots, A_n, A, C).$$

In other words, we can assume without loss of generality that A and C are among the points A_i. Let $I \subseteq \{1,\ldots,n\}$ be such that $i \in I \Leftrightarrow A_i \in ABC$, and set $J = \{1,\ldots,n\} \setminus I$ (thus $i \in J \Leftrightarrow A_i \in ACD \setminus AC$). Then we have concatenating chains realizing the minima and using Theorem 8

$$\mathbf{h}(A_1,\ldots,A_n,A,C) \leqslant \mathbf{c}_{AC}(A_i\ ;\ i \in I) + \mathbf{c}_{CA}(A_i\ ;\ i \in J) \leqslant 2|AC|^2 = \mathbf{h}(A,C).$$

b. Let E be the orthogonal projection of B on AC and F the orthogonal projection of D on AC. If we have equality, then we must have $\mathbf{c}_{AC}(A_i\ ;\ i \in I\} = \mathbf{c}_{CA}(A_i\ ;\ i \in J\} = |AC|^2$, hence by the case of equality in Theorem 8, the support $\{A_i\ ;\ i \in I\}$ is included in $\{A,B,C,E\}$ and the support $\{A_i\ ;\ i \in J\}$ is empty or reduced to $\{D\}$.

If $E \neq F$ then none of the A_i, $i \in I$, can be E, because one of the unsigned angles $|\widehat{AED}|$ or $|\widehat{CED}|$ is obtuse. Hence all the A_i are concentrated at two, three, or four vertices of $\mathcal{R}$, and at least two of these vertices are opposite.

If $E = F$, i.e. if $\mathcal{R}$ is a square, then the case that some of the A_i are in E may occur, but in that case all the four vertices are occupied. □

Corollary 10. *Conjecture 2 is true if three or four of the points A_i contain all points in their convex hull.*

Proof. We give the proof for four points; the case of three points is easier. Let A,B,C,D denote these points, in convex position and ordered counterclockwise. Without loss of generality, we assume that $\widehat{CBA} \geqslant \pi/2$ and that $\widehat{DCA} \geqslant \widehat{DAC}$. We consider two cases.

If $\widehat{DCA} \leqslant \pi/2$, then we split the quadrilateral $ABCD$ in three right or obtuse triangles ABC, $CD'D$ and $DD'A$, where D' is the orthogonal projection of D on AC. Let

$$\begin{aligned}
I_1 &= \big\{i \in \{1,\ldots,n\}\ ;\ A_i \in ABC\big\},\\
I_2 &= \big\{i \in \{1,\ldots,n\}\ ;\ A_i \in CD'D \setminus CD'\big\} \text{ and}\\
I_3 &= \{1,\ldots,n\} \setminus (I_1 \cup I_2).
\end{aligned}$$

Using Theorem 8, we obtain $\mathbf{c}_{AC}(A_i\ ;\ i \in I_1) \leqslant |AC|^2$, $\mathbf{c}_{CD}(A_i\ ;\ i \in I_2) \leqslant |CD|^2$ and $\mathbf{c}_{DA}(A_i\ ;\ i \in I_3) \leqslant |DA|^2$, hence $\mathbf{h}(A_1,\ldots,A_n) \leqslant \mathbf{h}(A,C,D)$.

If $\widehat{DCA} > \pi/2$, then we use that the triangle ACD is obtuse in C and we split the quadrilateral in two right or obtuse triangles ABC and DCA. With $I_1 = \big\{i \in \{1,\ldots,n\}\ ;\ A_i \in ABC\big\}$ and $I_2 = \{1,\ldots,n\} \setminus I_1$, we obtain by Theorem 8

$$\mathbf{c}_{AC}(A_i\ ;\ i \in I_1) \leqslant |AC|^2 \text{ and } \mathbf{c}_{DA}(A_i\ ;\ i \in I_2) \leqslant |DA|^2,$$

hence

$$\mathbf{h}(A_1,\ldots,A_n) \leqslant \mathbf{c}_{AC}(A_i\ ;\ i \in I_1) + |CD|^2 + \mathbf{c}_{DA}(A_i\ ;\ i \in I_2) \leqslant \mathbf{h}(A,C,D) < \mathbf{h}(A,D).$$

Now the smallest disk containing the n-tuple contains two or three of the points A,C,D in its boundary and no other A_i. □

Surprisingly we were not able to confirm Conjecture 3 for all quadrilaterals.

Corollary 11. *Conjecture 1 is true modulo a factor 2: Among any n-tuple* $(A_1, \ldots, A_n)$, $n > 3$, *there are two or three points* A_i, A_j, A_k *(with possibly* $i = j$*) such that*

$$\mathbf{h}(A_1, \ldots, A_n) \leqslant 2\mathbf{h}(A_i, A_j, A_k).$$

Proof. Let $\mathcal{D}$ be the smallest disk containing all the A_i and let $\mathcal{K}$ be any circumscribing square of $\mathcal{D}$. There are two or three points among the A_i, say A_j, A_k, A_ℓ with possibly $j = k$, which are on the boundary of $\mathcal{D}$ and which contain the center of $\mathcal{D}$ in their convex hull. We have $\mathbf{h}(A_j, A_k, A_\ell) \geqslant 2d^2$, where d is the diameter of $\mathcal{D}$: Indeed, if B is on the boundary of $\mathcal{D}$ such that A_jB is a diameter of it, then $\mathbf{h}(A_j, A_k, A_\ell) = \mathcal{C}\big([A_jA_kA_\ell]\big) \geq \mathcal{C}\big([A_jA_kBA_\ell]\big) = 2d^2$. We conclude using that $\mathbf{h}(A_1, \ldots, A_n) \leqslant 4d^2$ by Corollary 9. □

The following result states that space filling curves cannot help prove Property **P** for sets larger than right triangles, see the comments in Introduction. In its proof, a construction of Pólya's curve for a given right triangle is derived without going into details.

Theorem 12. *Suppose* Δ *is a right triangle with hypothenuse* AB *and that* $\Delta \subseteq \mathcal{N}$. *Suppose there exists a surjective mapping* $f : [0, 1] \to \mathcal{N}$ *such that* $f(0) = A, f(1) = B$ *and*

$$|f(t) - f(s)|^2 \leq |AB|^2|t - s| \quad \textit{for} \quad 0 \leq s, t \leq 1. \tag{15}$$

Then $\Delta = \mathcal{N}$.

Proof. Observe that f is continuous by (15). Without loss of generality, we can assume that $|AB| = 1$. Suppose that C is the third vertex of Δ and that $s_0 \in [0, 1]$ with $f(s_0) = C$. Then $s_0 \geq |AC|^2$ and $1 - s_0 \geq |CB|^2$ by (15). Since $|AC|^2 + |CB|^2 = |AB|^2 \leq 1$, this implies $s_0 = |AC|^2$.

Next let H denote the projection of C onto AB. We want to determine two values s_1^1 and s_1^2, $s_1^1 < s_0 < s_1^2$ such that $f(s_1^i) = H$. Observe that $f(s)$ must be in the disk $\mathcal{D}_{BC}$ of diameter BC if $s_0 \leqslant s \leqslant 1$ because by (15) we have

$$|Bf(s)|^2 + |Cf(s)|^2 \leqslant 1 - s + s - s_0 = 1 - s_0 = |BC|^2.$$

Now we can choose a sequence P_n in $\Delta \setminus \mathcal{D}_{BC}$ tending to H and t_n with $f(t_n) = P_n$. As shown before, we must have $0 \leq t_n < s_0$.

With $\delta_n = |HP_n|$ and $s_1^1 = |AH|^2$ we find on the one hand using (15)

$$t_n \geq |AP_n|^2 \geq (|AH| - \delta_n)^2 \geq s_1^1 - 2\delta_n|AH|,$$

on the other hand again using (15)

$$s_0 - t_n \geq |CP_n|^2 \geq (|CH| - \delta_n)^2 \geq s_0 - s_1^1 - 2\delta_n|CH|$$

and hence $t_n \leq s_1^1 + 2\delta_n|CH|$. This shows that t_n tends to s_1^1 as $\delta_n \to 0$, i.e. $P_n \to H$. By the continuity of f, we find $f(s_1^1) = \lim f(t_n) = H$. In a symmetric way, we prove that $f(s_1^2) = H$ where $s_1^2 = 1 - |BH|^2 \in [s_0, 1]$.

Continuing in this way to the projections H_0, H_1 of H onto AC and BC, respectively, and to their projections etc., we find a dense subset of $[0, 1]$ mapped onto a dense subset of Δ, on which f is uniquely determined. By the continuity of f, it is uniquely determined, we have $f([0, 1]) \subseteq \Delta$ and thus $\Delta = \mathcal{N}$ by the assumed surjectivity. □

4 The half-disk and the disk

We want to prove Conjecture 2 in the very special case of six points, five of which are in convex position and two of which form a diameter of the smallest disk containing them. For this, we fix A, B in the plane and will need that a half-disk of diameter AB satisfies Property $\mathbf{P}_3$. As a warm-up, we first prove that such a half-disk has Property $\mathbf{P}_2$.

Proposition 13. *Let $\mathcal{H}$ be a half-disk of diameter AB. Then for all $C, D \in \mathcal{H}$ one has $\mathbf{c}_{AB}(C,D) \leqslant |AB|^2$, with equality if and only if:*

- either the support $\{A,B,C,D\}$ is of cardinality at most 3 and is included in the half-circle $\overset{\frown}{AB}$ bounding $\mathcal{H}$,

- or, among C and D, one point belongs to $\overset{\frown}{AB}$ and the other one is its orthogonal projection on AB.

Proof. Let C' and D' be the orthogonal projections of C and D on AB. Without loss of generality, we assume that the points are ordered A, C', D', B on the line AB (possibly $C' = D'$), and that $|CC'| \leqslant |DD'|$. Then we have $|AC|^2 + |CD|^2 = |AD|^2 - 2\overrightarrow{AC}.\overrightarrow{CD} \leqslant |AD|^2$ and $|AD|^2 + |DB|^2 = |AB|^2 - 2\overrightarrow{AD}.\overrightarrow{DB} \leqslant |AB|^2$, hence

$$\mathbf{c}_{AB}(C,D) = |AC|^2 + |CD|^2 + |DB|^2 \leqslant |AB|^2. \tag{16}$$

Moreover, equality in (16) occurs if and only if $\overrightarrow{AC}.\overrightarrow{CD} = \overrightarrow{AD}.\overrightarrow{DB} = 0$, i.e.

- either $C \in \{A, D\}$ and $AD \perp DB$ (or $D \in \{A, B\}$), i.e. D is on the half-circle $\overset{\frown}{AB}$,

- or $AC \perp CD$ (hence $C = C' = D'$) and D is on $\overset{\frown}{AB}$. □

Remark. A half-disk $\mathcal{H}$ of diameter AB is maximal for inclusion among subsets satisfying $\mathbf{P}_2$: If a subset $\mathcal{K}$ of the plane contains $\mathcal{H}$ and satisfies $\mathbf{P}_2$, then $\mathcal{K} = \mathcal{H}$.

Such a half-disk is also the largest element among subsets satisfying $\mathbf{P}_2$ and entirely on one side of the straight line (AB): Any subset of a closed half-plane bounded by (AB) and satisfying $\mathbf{P}_2$ is contained in the corresponding half-disk of diameter AB.

An interesting question would be to describe *all* subsets satisfying this relative maximality property for $\mathbf{P}_2$. In particular we did not find such a maximal subset for $\mathbf{P}_2$ which also satifies $\mathbf{P}_3$, except the above half-disks.

We will go one step further and prove that a half-disk of diameter AB satisfies Property $\mathbf{P}_3$. To this purpose we will use the following lemma, which is of interest by itself.

Lemma 14. *Let $\mathcal{H}$ be a half-disk of diameter AB.*

a. *For every point D on the segment AB, different from A and B, there exists a unique pair of points C, E on the half-circle $\overset{\frown}{AB}$ bounding $\mathcal{H}$ such that $AE \perp CD$ and $BC \perp DE$.*

Likewise, for every $E \in \overset{\frown}{AB} \setminus \{A, B\}$ there exists a unique pair $(C, D) \in \overset{\frown}{AB} \times AB$ such that $AE \perp CD$ and $BC \perp DE$.

Such a polygonal line $[ACDEB]$ will be called an $\mathcal{M}$ in the rest of the article.

b. *If $[ACDEB]$ is an $\mathcal{M}$, then $\mathbf{c}_{AB}(C,D,E) = |AB|^2$.*

Remark. Since $AE \perp CD$, one has $|AC|^2+|DE|^2 = |AD|^2+|CE|^2$: Indeed, if I denotes the intersection point of AE and CD and a, c, d, e denote the distances $|AI|$, $|CI|$, $|DI|$, $|EI|$, respectively, then both sides are equal to $a^2+c^2+d^2+e^2$. This implies that the polygonal line $[ACDE]$ has the same cost as $[ADCE]$. The same holds for the lines $[CDEB]$ and $[CEDB]$. As a consequence, the minimum $\mathbf{c}_{AB}(C, D, E)$ is reached by the three polygonal lines $[ADCEB]$, $[ACDEB]$ and $[ACEDB]$.

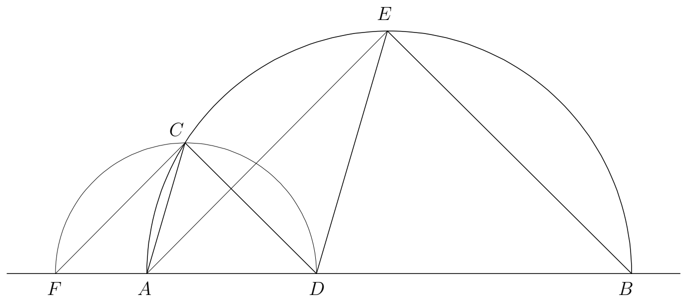

Figure 1: Points satisfying the conditions of Lemma 14

Proof. **a.** Observe that the conditions $AE \perp CD$ and $BC \perp DE$ are equivalent to $BE \parallel CD$ and $AC \parallel DE$ because the angles $\widehat{ACB}$ and $\widehat{AEB}$ are right angles by Thales' Theorem.

We begin with proving uniqueness for the first statement. Suppose that $BE \parallel CD$ and $AC \parallel DE$. Then let F denote the intersection of the straight line (AB) and the parallel to AE through C. As all of their corresponding edges and diagonals are parallel, the polygons $ADEB$ and $FACD$ are similar. Hence $|FA|/|AD| = |AD|/DB|$ which determines F uniquely. Moreover the angle $\widehat{FCD}$ is a right angle as $\widehat{AEB}$ is, and therefore C is on the half-circle $\overset{\frown}{FD}$. This determines C uniquely. Then E is determined as the unique point on $\overset{\frown}{AB}$ such that $DE \parallel AC$.

Given A, D, B, clearly the points F, C, E constructed as above give similar polygons $ADEB$ and $FACD$ and therefore $BE \parallel CD$ and $AC \parallel DE$.

Now, as D moves continuously and strictly monotonically from A to B on the segment AB, C and E move continuously and strictly monotonically from A to B on the arc $\overset{\frown}{AB}$. This follows from the above construction. This proves the second statement.

b. Put $u = \overrightarrow{AC}$, $v = \overrightarrow{CD}$, $w = \overrightarrow{DE}$ and $x = \overrightarrow{EB}$. Since the triangles ACD and DEB are similar, there exists $\lambda \in \mathbb{R}$ such that $w = \lambda u$ and $x = \lambda v$ and hence $v.w = u.x$. One has

$$|AB|^2 = \|u+v+w+x\|^2 = \|u\|^2+\|v\|^2+\|w\|^2+\|x\|^2+2\big(u.(v+w+x)+v.w+v.x+w.x\big),$$

but $u.(v+w+x) = \overrightarrow{AC}.\overrightarrow{CB} = 0$ and $v.w+v.x+w.x = (u+v+w).x = \overrightarrow{AE}.\overrightarrow{EB} = 0$, thus $|AB|^2 = \|u\|^2+\|v\|^2+\|w\|^2+\|x\|^2 = \mathbf{c}_{AB}(C, D, E)$. □

Theorem 15. *A half-disk $\mathcal{H}$ of diameter AB has Property $\mathbf{P}_3$. In particular, let $C, D, E \in \mathcal{H}\setminus\{A, B\}$ be three distinct points, such that their orthogonal projections on AB are ordered*

A, C', D', E', B. Then $\mathbf{c}_{AB}(C, D, E) \leqslant |AB|^2$, with equality if and only if $[ACDEB]$ is an $\mathcal{M}$.

Remark: The cases $\mathrm{card}\{A, B, C, D, E\} < 5$ are taken care of in Proposition 13.

It seems natural that the worst case is when the three polygonal lines $[ADCEB]$, $[ACDEB]$ and $[ACEDB]$ have the same cost, i.e. when $[ACDEB]$ is an $\mathcal{M}$, in which case we have equality by Lemma 14. Curiously, we haven't found a simple proof of this result. For this reason we give the lengthy proof in an appendix.

We conjecture that a half-disk has not only Property $\mathbf{P}_3$:

Conjecture 5. *A half-disk $\mathcal{H}$ of diameter AB satisfies Property $\mathbf{P}$. Moreover, if points $A_1, \ldots, A_n$ belong to $\mathcal{H}$ and are such that $\mathbf{c}_{AB}(A_1, \ldots, A_n) = 2|AB|^2$ then the support $\mathcal{S} = \{A, B, A_1, \ldots, A_n\}$ has cardinality at most 5 and either $\mathcal{S} = \{A, B\}$, or $\mathcal{S} = \{A, B, C\}$ with $C \in \widehat{AB}$, or $\mathcal{S} = \{A, B, C, D\}$ with $C \in \widehat{AB}$ and D is the orthogonal projection of C on (AB), or the five points form an $\mathcal{M}$.*

Theorem 15 immediately yields the following

Corollary 16. *Conjecture 2 is true for six points in a disk, two of which form a diameter.*

Proof. Let AB be a diameter of a disk $\mathcal{D}$ and $C, D, E, F \in \mathcal{D}$ be further points. If all the four points C, D, E, F are in the same (closed) half-disk of diameter AB then, by Proposition 13, one has $\mathbf{h}(A, B, C, D, E, F) \leqslant \mathbf{c}_{AB}(C, D) + \mathbf{c}_{BA}(E, F) \leqslant 2|AB|^2$. It is the same if the four points C, D, E, F are in pairs on each half-disk. If one point, say F, is in one half-disk and the three others in the other one then one has $\mathbf{h}(A, B, C, D, E, F) \leqslant \mathbf{c}_{AB}(C, D, E) + \mathbf{c}_{BA}(F) \leqslant 2|AB|^2$ by (5) and Theorem 15. □

Another result concerning disks is the following.

Theorem 17. *Consider $n \geq 5$ and $A_1, \ldots, A_n$ on a circle $\mathcal{K}$ with center O. Then Conjecture 2 holds for $\{A_1, \ldots, A_n, O\}$.*

Proof. If $A_1, \ldots, A_n$ are contained in a half-circle of $\mathcal{K}$ then $\{A_1, \ldots, A_n, O\}$ are in a convex position and Conjecture 2 holds in this case by Proposition 1.

Now we assume that O is in the convex hull of $A_1, \ldots, A_n$. Therefore we can choose A_i, A_j, A_k such that O is in the triangle $A_i A_j A_k$ and, as indicated in the proof of Corollary 11, we have $\mathbf{h}(A_i, A_j, A_k) \geq 8r^2$, where r denotes the radius of $\mathcal{K}$.

We can assume that $A_1, \ldots, A_n$ are ordered. Put $\varphi_i = \widehat{A_i O A_{i+1}}$ with $A_{n+1} = A_1$. If any of the $\varphi_i \geq \frac{\pi}{2}$, say φ_n, then

$$\mathbf{h}(A_1, \ldots, A_n, O) \leqslant \mathcal{C}\big([A_1 \cdots A_n O A_1]\big) \leq \mathcal{C}\big([A_1 \cdots A_n A_1]\big)$$

and we can conclude with Proposition 1.

So we can assume from now on that all the $\varphi_i < \frac{\pi}{2}$. Finally, we can assume without loss of generality that φ_n is the largest of them. Of course $\varphi_1 + \ldots + \varphi_n = 2\pi$.
Now we calculate $|A_i A_{i+1}| = 2r \sin\left(\frac{\varphi_i}{2}\right)$ and obtain

$$\mathbf{h}(A_1, \ldots, A_n, O) \leqslant \mathcal{C}\big([A_1 \cdots A_n O A_1]\big) = 2r^2 + \sum_{i=1}^{n-1} 4r^2 \sin^2\left(\frac{\varphi_i}{2}\right). \tag{17}$$

It is sufficient to show that the right hand side is at most $8r^2$, or equivalently

$$\sum_{i=1}^{n-1} 2\sin^2\left(\frac{\varphi_i}{2}\right) \leq 3. \tag{18}$$

Now $2\sin^2(\frac{x}{2}) = 1 - \cos x$ is convex on the interval $[0, \frac{\pi}{2}]$ and we can use that

$$f(x) + f(y) \leq f(a) + f(b), \text{ if } f \text{ is convex on } [a,b], x, y \in [a,b] \text{ and } x + y = a + b. \tag{19}$$

Let m denote the unique integer such that $m\varphi_n \leq 2\pi < (m+1)\varphi_n$. Since $\frac{2\pi}{n} \leq \varphi_n < \frac{\pi}{2}$, we have $4 \leq m \leq n$. We now use that $\varphi_1 + \ldots + \varphi_{n-1} = 2\pi - \varphi_n$ and write $2\pi - \varphi_n = (m-1)\varphi_n + r_n + (n-m)\cdot 0$, where $r_n = 2\pi - m\varphi_n \in [0, \varphi_n[$.

We use (19) repeatedly in the following way. If two of the φ_i are strictly between 0 and φ_n, say φ_1 and φ_2, then we put $a = 0$, $b = \varphi_1 + \varphi_2$ if $\varphi_1 + \varphi_2 \leq \varphi_n$, but $a = \varphi_1 + \varphi_2 - \varphi_n$, $b = \varphi_n$ if $\varphi_1 + \varphi_2 \geq \varphi_n$. In both cases, we have $f(\varphi_1) + f(\varphi_2) \leq f(a) + f(b)$ and therefore

$$\sum_{i=1}^{n-1} 2\sin^2\left(\frac{\varphi_i}{2}\right) \leq \sum_{i=1}^{n-1} 2\sin^2\left(\frac{\tilde{\varphi}_i}{2}\right),$$

where $\tilde{\varphi}_1, \ldots \tilde{\varphi}_{n-1}$ denote $a, b, \varphi_3, \ldots, \varphi_{n-1}$ in a different way. Observe that at most $n-2$ of the $\tilde{\varphi}_i$ are strictly between 0 and φ_n. This procedure can be repeated with the $\tilde{\varphi}_i$, etc. until we have

$$\sum_{i=1}^{n-1} 2\sin^2\left(\frac{\varphi_i}{2}\right) \leq \sum_{i=1}^{n-1} 2\sin^2\left(\frac{\tilde{\varphi}_i}{2}\right),$$

where $\sum_{i=1}^{n-1} \tilde{\varphi}_i = 2\pi - \varphi_n$, $0 \leq \tilde{\varphi}_i \leq \varphi_n$ and at most one of the $\tilde{\varphi}_i$ is strictly between 0 and φ_n. By the choice of m, we then have that $m-1$ of the $\tilde{\varphi}_i$ equal φ_n, $n-m$ of them vanish and the remaining one equals r_n.

Thus we have shown that

$$\sum_{i=1}^{n-1} 2\sin^2\left(\frac{\varphi_i}{2}\right) \leq 2\sin^2\left(\frac{r_n}{2}\right) + 2(m-1)\sin^2\left(\frac{\varphi_n}{2}\right). \tag{20}$$

It remains to estimate $g(\varphi) = 2(m-1)\sin^2\left(\frac{\varphi}{2}\right) + 2\sin^2\left(\frac{2\pi - m\varphi}{2}\right)$ on the interval $I =]\frac{2\pi}{m+1}, \frac{2\pi}{m}]$. By the convexity of $f(x) = 2\sin^2(\frac{x}{2})$ on $[0, \varphi_n]$, the maximum of g can only be attained at the boundaries of I and we have that

$$\sum_{i=1}^{n-1} 2\sin^2\left(\frac{\varphi_i}{2}\right) \leq \max\left(2m\sin^2\left(\frac{\pi}{m+1}\right), 2(m-1)\sin^2\left(\frac{\pi}{m}\right)\right). \tag{21}$$

Thus it remains to show that $2(m-1)\sin^2\left(\frac{\pi}{m}\right) \leq 3$ for $m \geqslant 4$. For $m = 4$ we have equality; for $m = 5$ we obtain $8\sin^2\left(\frac{\pi}{5}\right) = 5 - \sqrt{5} \leqslant 3$; for $m = 6$ we have $10\sin^2\left(\frac{\pi}{6}\right) = \frac{5}{2} \leqslant 3$. For larger m, we use that $\sin x \leq x$ for nonnegative x. Therefore it is sufficient to have $\frac{2\pi^2}{m} \leq 3$, which is the case if $m \geqslant 7$. $\square$

Appendix: Proof of Theorem 15

Let A, B, C, D, E be as in the statement, i.e., with the projections on AB ordered A, C', D', E', B. By compactness of $\mathcal{H}^3$ and by continuity of the map $(C, D, E) \mapsto \mathbf{c}_{AB}(C, D, E)$, there exists (at least) a configuration (C, D, E) that realizes the maximum of $\mathbf{c}_{AB}(C, D, E)$ (possibly with coinciding points).

By contradiction, assume that $\mathbf{c}_{AB}(C, D, E) > |AB|^2$. By Proposition 13 the five points must be all distinct, and by Theorem 8 the convex hull $ABCDE$ is not a triangle, because this triangle would be right or obtuse. As a consequence, among C, D and E, at least two of them are extremal points of $ABCDE$.

We first prove that, for this worst configuration, the three unsigned angles $|\widehat{ACD}|$, $|\widehat{EDC}|$, and $|\widehat{DEB}|$ are less than a right angle.

If the angle $|\widehat{ACD}|$ equals or exceeds a right angle, then we would have

$$\mathbf{c}_{AB}(C, D, E) \leqslant \mathcal{C}\big([ACDEB]\big) \leqslant \mathcal{C}\big([ADEB]\big) \leqslant \max\big\{\mathcal{C}\big([ADB]\big), \mathcal{C}\big([AEB]\big)\big\} \leqslant |AB|^2,$$

contradicting our assumption. The third inequality above comes from the fact that one of the angles $|\widehat{DEB}|$ (if $|EE'| \leqslant |DD'|$) or $|\widehat{ADE}|$ (if $|DD'| \leqslant |EE'|$) is at least $\pi/2$. The cases $|\widehat{EDC}| \geqslant \pi/2$ and $|\widehat{DEB}| \geqslant \pi/2$ are similar.

It follows that $|DD'| < \max\big(|CC'|, |EE'|\big)$, hence C and E are extremal points of $ABCDE$. In particular there exists a straight line d passing through C such that all four points A, B, D, E are on the same side of d. Now the straight line orthogonal to d and passing through C meets the half-circle $\overset{\frown}{AB}$ at some point $\widetilde{C}$ satisfying $|\widetilde{C}X| \geqslant |CX|$ for all $X \in \{A, B, D, E\}$, with (at least) one equality only for $\widetilde{C} = C$, hence $\mathbf{c}_{AB}(C, D, E) \leqslant \mathbf{c}_{AB}(\widetilde{C}, D, E)$, with equality if and only if $\widetilde{C} = C$. In other words, in a worst configuration, C is on the half-circle $\overset{\frown}{AB}$; the same holds for E.

Set

$$\mathcal{C}_1 = \mathcal{C}\big([ADCEB]\big),\ \mathcal{C}_2 = \mathcal{C}\big([ACDEB]\big),\ \mathcal{C}_3 = \mathcal{C}\big([ACEDB]\big).$$

Thus we have $\mathbf{c}_{AB}(C, D, E) = \min(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

We now prove that D is on the segment AB. Indeed, let D' denote the orthogonal projection of D on AB and let $\mathcal{C}'_i$ denote the quantity analogous to $\mathcal{C}_i$ with D' instead of D. Then we claim that $\mathcal{C}'_i \geqslant \mathcal{C}_i$, with one or more equalities if and only if $D = D'$. For $i = 2$, this comes from $|CD'| \geqslant |CD|$ and $|D'E| \geqslant |DE|$, yielding $\mathcal{C}\big([CDE]\big) \leqslant \mathcal{C}\big([CD'E]\big)$, with equality only if $D = D'$. For $i = 1$, this comes from $|DD'| < |CC'|$ and, using (6), $\mathcal{C}'_1 - \mathcal{C}_1 = 2\big(|ID'|^2 - |ID|^2\big) \geqslant 0$, where I is the midpoint of A and C, with equality only for $D = D'$. The case $i = 3$ is similar.

At this point, several cases can occur: The identity $\mathbf{c}_{AB}(C, D, E) = \min(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ can be reached by one, two, or all three values of the $\mathcal{C}_i$.

If $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$, then we obtain that $AE \perp CD$ and $BC \perp DE$, i.e., the configuration is an $\mathcal{M}$, yielding $\mathbf{c}_{AB}(C, D, E) = |AB|^2$ by Lemma 14, a contradiction.

If $\mathcal{C}_1 < \min(\mathcal{C}_2, \mathcal{C}_3)$, then we slightly move D in the direction B. This slightly increases both AD and DC, hence $\mathcal{C}_1$ slightly increases, and $\mathcal{C}_2$ and $\mathcal{C}_3$ change only slightly, hence $\mathbf{c}_{AB}(C, D, E)$ slightly increases, contradicting the maximality of $\mathbf{c}_{AB}(C, D, E)$. In the same way, one cannot have $\mathcal{C}_3 < \min(\mathcal{C}_1, \mathcal{C}_2)$.

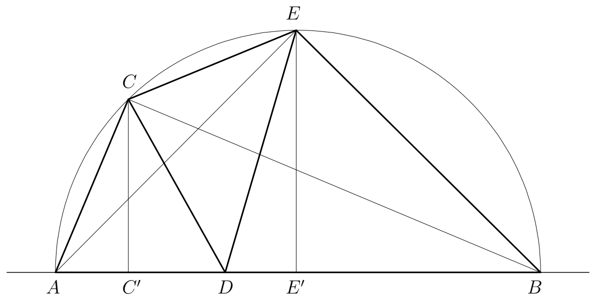

Figure 2: A current configuration of the five points

If $\mathcal{C}_2 < \min(\mathcal{C}_1, \mathcal{C}_3)$ then we also move D to reach the contradiction, but the move depends on the respective positions of the points: Let J be the midpoint of CE, and J' its orthogonal projection on AB. If D is between A and J', then by slightly moving D towards A we increase the distance $|DJ|$, hence we slightly increase $\mathcal{C}([CDE])$ by (6). In this manner $\mathcal{C}_2$ slightly increases, $\mathcal{C}_1$, $\mathcal{C}_3$ slightly change and once again $\mathbf{c}_{AB}(C, D, E)$ increases, a contradiction. If D is between J' and B, then we move D towards B. If $D = J'$, both directions increase $\mathbf{c}_{AB}(C, D, E)$.

As a consequence, in a worst configuration, exactly two of the values $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are equal and the third one is greater. The cases $\mathcal{C}_1 = \mathcal{C}_2 < \mathcal{C}_3$ and $\mathcal{C}_2 = \mathcal{C}_3 < \mathcal{C}_1$ are symmetric and will be treated afterwards.

We now treat the case $\mathcal{C}_1 = \mathcal{C}_3 < \mathcal{C}_2$ and we will prove that $\mathbf{c}_{AB}(C, D, E) \leqslant |AB|^2$, yielding a contradiction in this case. Without loss of generality, we can assume that $|AD| \leqslant |DB|$. Observe that $\mathcal{C}_1 < \mathcal{C}_2$ implies

$$\overrightarrow{AE}.\overrightarrow{DC} > 0. \tag{22}$$

We have $|AB|^2 - \mathcal{C}_1 = 2\big(\overrightarrow{AD}.\overrightarrow{DC} + \overrightarrow{CE}.\overrightarrow{EB}\big)$ and $|AB|^2 - \mathcal{C}_3 = 2\big(\overrightarrow{ED}.\overrightarrow{DB} + \overrightarrow{AC}.\overrightarrow{CE}\big)$, and these quantities are equal. We also have $\overrightarrow{AC}.\overrightarrow{CE} = \overrightarrow{CE}.\overrightarrow{EB}$, because $\overrightarrow{AC} + \overrightarrow{BE} = 2\overrightarrow{OJ}$, where O is the midpoint of AB and J the midpoint of CE, and $\overrightarrow{OJ} \perp \overrightarrow{CE}$. It follows that

$$\overrightarrow{AD}.\overrightarrow{DC} = \overrightarrow{ED}.\overrightarrow{DB} \tag{23}$$

Let us fix coordinates on the straight line (AB), with origin O (the midpoint of AB), such that the abscissa of A is -1, hence the abscissa of B is 1. Let x, d, y denote the abscissae of C, D, E respectively. In this manner we have $\overrightarrow{AD}.\overrightarrow{DC} = (1+d)(x-d)$ and $\overrightarrow{ED}.\overrightarrow{DB} = (d-y)(1-d)$. Then (23) implies

$$y = ax + b, \quad \text{with} \quad a = -\frac{1+d}{1-d} \quad \text{and} \quad b = \frac{2d}{1-d}. \tag{24}$$

Now Lemma 14 provides a unique pair of points C^* and E^*, of abscissae denoted by x^* and y^* respectively, such that $AE^* \perp C^*D$ and $BC^* \perp DE^*$.

Because of (22) we have $x^* < x$: Indeed, if x increases to $\tilde{x}$ then y decreases to $\tilde{y}$ by (24). Hence the angle $\widehat{CD\tilde{C}} < 0$ whereas $\widehat{EA\tilde{E}} > 0$. So the angle between $\overrightarrow{AE}$ and $\overrightarrow{DC}$

is smaller than the one between $\overrightarrow{AE}$ and $\overrightarrow{DC}$. As the angle between $\overrightarrow{AE^*}$ and $\overrightarrow{DC^*}$ is a right angle, the one between $\overrightarrow{AE}$ and $\overrightarrow{DC}$ is smaller than a right angle if $x > x^*$ and larger than a right angle if $x < x^*$. This means that $\overrightarrow{AE}.\overrightarrow{DC} > 0$ if $x > x^*$ and < 0 if $x < x^*$. Therefore (22) implies $x > x^*$.

Set $f(x) = |AB|^2 - \mathcal{C}_1$, with y given by (24). Since we have assumed $|AD| \leqslant |DB|$, this function is defined at least in the interval $]-1, d]$. We have $f(d) = 0$ by Proposition 13 and $f(x^*) = 0$ by Lemma 14. It remains to prove that $f(x) > 0$ if $x^* < x < d$. For any $u \in [-1, 1]$, we use the notation $r(u) = \sqrt{1-u^2}$. We have

$$\begin{aligned}\frac{1}{2}f(x) &= \overrightarrow{AD}.\overrightarrow{DC} + \overrightarrow{AC}.\overrightarrow{CE}\\ &= (1+d)(x-d) + (1+x)(y-x) + r(x)\big(r(y) - r(x)\big)\\ &= -(1+d+d^2) + dx + y + xy + r(x)r(y),\end{aligned}$$

where y is given by (24). Hence we have $f(x) > 0$ if and only if

$$r(x)r(y) > \alpha x^2 + \beta x + \gamma, \tag{25}$$

where α, β, γ are the coefficients of $1 + d + d^2 - dx - y - xy$, considered as a polynomial of degree 2 in x. One finds $\alpha = -a = \dfrac{1+d}{1-d}$, $\beta = -d - a - b = 1 - d$, and $\gamma = 1 + d + d^2 - b = \dfrac{1-2d-d^3}{1-d}$. The inequality (25) will be satisfied if we prove that the polynomial

$$P(x) := \big(r(x)r(y)\big)^2 - (\alpha x^2 + \beta x + \gamma)^2$$

is positive on the open interval $]x^*, d[$. We already have $P(x^*) = P(d) = 0$. We also have $P(-1) < 0$ because $\lim_{x\to -1} f(x) = -|AD|^2 < 0$.

Denoting $P = a_0 + \cdots + a_4 x^4$, one finds $a_4 = a^2 - \alpha^2 = 0$ (i.e. P is of degree at most 3) and $a_3 = 2ab - 2\alpha\beta = 2a(b + 1 - d) = -2\dfrac{(1+d)(1+d^2)}{(1-d)^2} < 0$.

As a consequence we obtain $\lim\limits_{x\to-\infty} P(x) = +\infty$. Since $P(-1) < 0$, the third root of P is outside $[x^*, d]$, and we have $P > 0$ on $]x^*, d[$.

Now we treat the last case $\mathcal{C}_1 = \mathcal{C}_2 < \mathcal{C}_3$. We still use the same notations: $A = (-1, 0)$, $O = (0, 0)$, $B = (1, 0)$, $C = (x, r(x))$, $D = (d, 0)$, and $E = (y, r(y))$, with $-1 < x < d < y < 1$.

Let us fix E. Lemma 14 gives $C^* \in \overset{\frown}{AB}$ and $D^* \in AB$ unique, of abscissae denoted by x^* and d^* respectively, such that $AE \perp C^*D^*$ and $BC^* \perp D^*E$.

The assumption $\mathcal{C}_1 = \mathcal{C}_2$ gives $AE \perp DC$, i.e. $(y+1)(x-d) + r(x)r(y) = 0$, hence

$$d = x + \lambda r(x), \quad \text{with} \quad \lambda = \frac{r(y)}{1+y} = \frac{d^* - x^*}{r(x^*)}. \tag{26}$$

The last equality comes from $CD \parallel C^*D^*$.

The assumption $\mathcal{C}_1 = \mathcal{C}_2 < \mathcal{C}_3$ implies $-1 < x < x^*$ and $-1 < d < d^*$. The assumption $\mathcal{C}_2 < \mathcal{C}_3$ gives $\overrightarrow{CB}.\overrightarrow{DE} > 0$. This implies that the angle $\angle\big(\overrightarrow{CB}, \overrightarrow{DE}\big)$ is less than $\pi/2$. It follows that $-1 < x < x^*$ and $-1 < d < d^*$: Indeed, if $x < x_1$ and $d < d_1$ are such that $CD \parallel C_1D_1$ (in order to keep $\mathcal{C}_1 = \mathcal{C}_2$) then $\angle\big(\overrightarrow{CB}, \overrightarrow{DE}\big) < \angle\big(\overrightarrow{C_1B}, \overrightarrow{D_1E}\big)$.

As before, we put $f(x) = |AB|^2 - \mathcal{C}_1$ with d given by (26). We have $\lim_{x\to -1} f(x) = 0$, and by Lemma 14 we have $f(x^*) = 0$. It remains to prove that $f(x) > 0$ if $-1 < x < x^*$.

As before, we have

$$\frac{1}{2}f(x) = \overrightarrow{AD}.\overrightarrow{DC} + \overrightarrow{AC}.\overrightarrow{CE} = (1+d)(x-d) + (1+x)(y-x) + r(x)\big(r(y) - r(x)\big).$$

One calculates using (26):

$$f(x) = \frac{2}{1+y}\Big((1-y)x^2 + (y^2-1)x + (y-1)(y+2) + (y-x)r(x)r(y)\Big)$$

and we are done if we prove that

$$Q(x) := \big((y-x)r(x)r(y)\big)^2 - \big((1-y)x^2 + (y^2-1)x + (y-1)(y+2)\big)^2 > 0$$

for all $x \in \,]-1, x^*[\,$. We consider Q as a polynomial in x (of degree 4) and y as a parameter.

Since we have $Q(-1) = 0$ and also $Q(x) = 0$ if $y = 1$, it follows that $x+1$ and $y-1$ are factors of Q. One finds $Q(x) = 2(x+1)(y-1)R(x)$ with

$$R(x) = x^3 - (2+y)x^2 + y(y+2)x + 2 - 2y^2 - y^3 \tag{27}$$

and it remains to prove that $R < 0$ on $]-1, x^*[\,$. We know that $R(x^*) = 0$. The computation gives $R(-1) = -(1+y)^3 < 0$, $R(y) = 2(1-y^2) > 0$, and $R''(x) = 2\big(3x - (2+y)\big)$, which vanishes at $\frac{1}{3}(y+2) > y$. As a consequence, R is concave on $[-1, y]$. Since $x^* < y$ and $R(x^*) < R(y)$, there exists, by the mean value theorem, a c between x^* and y such that $R'(c) = \frac{R(y)-R(x^*)}{y-x^*} > 0$. Hence the function R', decreasing on $[1, y]$, is positive on $[1, c]$ and therefore R is increasing on $[-1, x^*]$. Since $R(x^*) = 0$, we have $R < 0$ on $]-1, x^*[\,$. $\square$

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