ON A METHOD TO DISPROVE GENERALIZED BRUNN–MINKOWSKI INEQUALITIES

NICOLAS JUILLET

ABSTRACT. We present a general method to disprove generalized Brunn–Minkowski inequalities. We initially developed this method in [14] in the particular case of the sub-Riemannian Heisenberg group in order to prove that this space does not satisfy a curvature-dimension condition in the sense of Lott–Villani and Sturm.

INTRODUCTION

New developments in analysis and geometry deal with a synthetic definition of Ricci curvature in the non-smooth context of metric spaces, whereas Ricci curvature originated in smooth Riemannian manifolds. Precisely, the property for a space to satisfy a so-called curvature-dimension condition $CD(K, N)$ is interpreted as behaving in some aspects as a Riemannian manifold with dimension $\leq N$ and Ricci curvature $\geq K$ at any point. Lott and Villani [18, 17] and independently Sturm [24, 25] managed to define a new notion of curvature-dimension $CD(K, N)$ using optimal transport, a tool that was traditionally used in probability and statistics. They exploited some nice aspects of this theory. Two of them are — (i) the theory can be developed on very general sets (typically on Polish metric spaces (X, ρ)), (ii) the geodesics of the Wasserstein space (a metric space made of the probability measures used in optimal transport) are represented as a probability measure in the space of the geodesics of (X, ρ). For details about geodesics (in the sense of minimizing curves) and curves in metric spaces see for instance [1, 5].

Up to now there are two concepts of families of curvature-dimension. The first family is connected to the correspondence with the curvature-dimension theory of Bakry and Émery [3] and provides results in diffusion semi-group theory, such as logarithmic Sobolev inequalities (see [2]). In the eighties, Bakry and Émery introduced the criterion $CD_{BE}(K, N)$

(the definition of which differs from that of $CD(K, N)$) in order to have a practical necessary condition for a Riemannian manifold with a diffusion operator to satisfy such inequalities. The corresponding results are preserved by the new definition $CD(K, N)$ (see [26]). We are more interested in the second family, which has a flavor of differential geometry. Lott, Sturm and Villani were able to prove theorems on the volume of balls, for instance the generalized Bishop–Gromov theorem or the generalized Bonnet–Myers theorem. In this paper we are especially interested in the geodesic Brunn–Minkowski inequality $BM_g(K, N)$, an inequality that is quite transversal in mathematics (see [4, 9, 22]) and that is a direct consequence of $CD(K, N)$. We will present a method to disprove this Brunn–Minkowski inequality or its variants. This can be thought of as a contribution to the classification of the spaces satisfying a curvature-dimension condition among all metric measure spaces. As the theory of Lott, Sturm and Villani is quite recent, this classification is not completely finished. Our method appeared in [14] in relation with the case of the first Heisenberg group.

This note is based on the talk "Synthetic Ricci curvature bounds in the Heisenberg group" in the first MSJ-SI in Kyoto, which explained our contributions in understanding the geometry of the Heisenberg group H_d [8, 14, 11, 12, 13]. Compared to the talk, in the article we concentrate on the central argument of [14] and present it in a more general framework including variations of the "Brunn–Minkowski inequality". I would like to thank the organizers of the first MSJ-SI very much for this wonderful and very interesting international conference.

1. The classical Brunn–Minkowski inequality

We state in this section the two equivalent forms of the classical Brunn–Minkowski inequality. Let A and B be two subsets of $\mathbb{R}^d$. The Minkowski sum is defined as $A + B = \{x + y \in \mathbb{R}^d \mid x \in A \text{ and } y \in B\}$. The Brunn–Minkowski inequality states

$$(1) \qquad (\mathcal{L}_d(A + B))^{1/d} \geq \mathcal{L}_d(A)^{1/d} + \mathcal{L}_d(B)^{1/d}$$

where $\mathcal{L}_d$ is the Lebesgue measure on $\mathbb{R}^d$. It is also satisfied if A and B are Borel sets. We know that $A + B$ is not always Borel but it is always measurable if A and B are Borel sets [7, 23]. For further details and more results, see [15].

Let $s \in [0, 1]$ and apply the Brunn–Minkowski inequality to $(1 - s)A' = \{(1 - s)x \in \mathbb{R}^d \mid x \in A'\}$ and $sB' = \{sy \in \mathbb{R}^d \mid y \in B'\}$. We obtain

$$(2) \qquad (\mathcal{L}_d((1 - s)A' + sB'))^{1/d} \geq (1 - s)\mathcal{L}_d(A')^{1/d} + s\mathcal{L}_d(B')^{1/d}$$

for any Borel sets A' and B' simply because we know the measure of the dilated sets. Notice that $(1-s)A'+sB' = \{(1-s)x+sy \in \mathbb{R}^d \mid x \in A', y \in B'\}$ is the set containing the convex combinations with ratio $(1-s) : s$ between the points of A' and the points of B'. Contrary to (1), this combination of A' and B' is purely affine (it is independent of the origin $0_{\mathbb{R}^d}$). Both inequalities are actually equivalent because (1) can be recovered from (2) by using $A = 2A'$, $B = 2B'$ and $s = 1/2$.

Several proofs are known for the Brunn–Minkowski inequality (see [9]). One of the more recent proofs, due to McCann, makes use of the optimal transport theory [19].

2. Generalized Brunn–Minkowski inequalities

From the Euclidean Brunn–Minkowski inequality there are at least two types of generalizations: the generalized multiplicative Brunn–Minkowski inequality BM_m and the generalized geodesic Brunn–Minkowski inequality BM_g. The first one occurs for a measure group $(G, \cdot, \mu)$ if you replace $+$ by $\cdot$ and $\mathcal{L}_d$ by μ in inequality (1). There is still an undefined parameter, namely the dimension d. Hence with these modifications, we will denote (1) by $BM_m(d)$. Since $(1+t)^\alpha \leq 1+t^\alpha$ for any $t \geq 0$ and $\alpha \in [0,1]$, if $BM_m(N)$ holds for $(G, \cdot, \mu)$ and if $N' \leq N$, the inequality $BM_m(N')$ is true as well. Once again $BM(N)$ is written

$$(3) \qquad (\mu(A \cdot B))^{1/N} \geq \mu(A)^{1/N} + \mu(B)^{1/N}$$

for any Borel sets A and B.

The second generalization is the geodesic Brunn–Minkowski inequality $BM_g(K, N)$. The easiest version $BM_g(0, N)$ that we will simply note $BM_g(N)$ is the generalization of (2) for $N = d$. For a geodesic metric measure space (X, ρ, μ) (see [5] for a definition), we parametrize the minimal curves with constant speed on $[0, 1]$. Let then $\mathcal{M}^s(x, y) = \{\gamma(s) \in X \mid (\gamma(0), \gamma(1)) = (x, y)$ and γ is a minimal curve$\}$ and call the points $\gamma(s)$ of this definition s-intermediate points ($\frac{1}{2}$ - intermediate points are midpoints). Then $BM_g(N)$ is the following inequality

$$(4) \qquad \mu\left(\mathcal{M}^s(A, B)\right)^{1/N} \geq (1-s)\mu(A)^{1/N} + s\mu(B)^{1/N}$$

for any $s \in [0, 1]$, $\mathcal{M}^s(A, B)$ is defined by $\bigcup_{(x,y) \in A \times B} \mathcal{M}^s(x, y)$ for any Borel sets. If $\mathcal{M}^s(A, B)$ is not measurable, replace in (4) $\mu(\mathcal{M}^s(A, B))$ by the outer measure of $\mathcal{M}^s(A, B)$. Note that (4) is a generalization of the Euclidean case (2) for $d = N$ because in $\mathbb{R}^d$ minimal curves are segments so that $\mathcal{M}^s(x, y)$ is the set $\{(1-s)x + sy\}$. The generalized versions of the geodesic Brunn–Minkowski inequalities have become popular since the optimal transport proof in the Euclidean case. This

proof can be adapted without any problems to Riemannian manifolds with Ricci curvature bounded from below [6, 25]. From the concavity of $t \to t^\alpha$ for $\alpha \in [0,1]$ we obtain that for a given geodesic metric measure space and $N' \geq N$, $BM_g(N')$ is a consequence of $BM_g(N)$.

The inequality $BM_g(K, N)$ is a generalization of $BM_g(N)$ that is also $BM_g(0, N)$. This is a natural generalization of the inequality that is satisfied by Riemannian manifolds of dimension $\leq N$ and Ricci curvature $\geq K$. We give the precise definition of $BM_g(K, N)$ for $N > 1$ that can be found in [25] (see also [26]). The property $BM_g(K, N)$ means that for two arbitrary Borel sets A and B and $s \in [0, 1]$

$$(5) \qquad \mu(\mathcal{M}^s(A, B))^{1/N} \geq \tau_{(1-s)}^{K,N}(\Theta)\mu(A)^{1/N} + \tau_s^{K,N}(\Theta)\mu(B)^{1/N}$$

where

$$\Theta = \begin{cases} \inf_{(x,y)\in A\times B} \rho(x, y), & \text{if } K \geq 0 \\ \sup_{(x,y)\in A\times B} \rho(x, y), & \text{if } K < 0 \end{cases}$$

and

$$\tau_s^{K,N}(\Theta) =$$
$$\begin{cases} +\infty, & \text{if } K\Theta^2 \geq (N-1)\pi^2 \\ s^{1/N}\left(\dfrac{\sin(s\Theta\sqrt{K/(N-1)})}{\sin(\Theta\sqrt{K/(N-1)})}\right)^{1-1/N}, & \text{if } 0 < K\Theta^2 < (N-1)\pi^2 \\ s, & \text{if } K\Theta^2 = 0 \\ s^{1/N}\left(\dfrac{\sinh(s\Theta\sqrt{-K/(N-1)})}{\sinh(\Theta\sqrt{-K/(N-1)})}\right)^{1-1/N}, & \text{if } K\Theta^2 < 0. \end{cases}$$

Here $\tau_s^{K,N}(\Theta)$ is exactly the normalized contraction rate of ratio s for a point at distance Θ in the model spaces of constant sectional curvature $K/(N-1)$ and dimension N (Euclidean space, scaled sphere, scaled hyperbolic space). Note that in this case K is the best lower bound for the Ricci curvature.

Other generalizations have been considered in the literature. See [9] for some of them. Typically, they state that a set $\mathbf{M}(A, B)$ obtained as a combination of two sets A and B has μ-measure greater than a function of the μ-measures of A and B or sets related to them. Note however that for $BM_g(K, N)$ with $K \neq 0$ this right-hand side is a little more intricate. In the next section we give an effective way to bound $\mathbf{M}(A, B)$ from above and therefore disprove some generalized Brunn–Minkowski inequality in a given metric measure space.

3. Estimating the Minkowski combination

We state our method in the form of a theorem.

Theorem 1. *Let $U, V \subset \mathbb{R}^d$ be open sets, $\mathbf{M} : V \times U \to \mathbb{R}^d$ a smooth map and a, b, o three points such that $\mathbf{M}(a, b) = o$. Let $\mathbf{I}$ be a smooth map from V to U such that $\mathbf{I}(b) = a$, $\mathbf{I}(V) = U$ and $\mathbf{M}(\mathbf{I}(p), p) = o$ for $p \in U$. Let $B_r \subset U$ be a Euclidean ball of center b and radius r and $A_r = \mathbf{I}(B_r)$. Then we have*

$$\limsup_{r \to 0^+} \frac{\mathcal{L}_d(\mathbf{M}(A_r, B_r))}{\mathcal{L}_d(B_r)} \leq 2^d \operatorname{Jac}(\mathbf{M_a})(b),$$

where $\mathbf{M}_a$ is the map $p \to \mathbf{M}(a, p)$. Moreover for $\mu = f\mathcal{L}^d$ with continuous and positive density f,

$$\limsup_{r \to 0^+} \frac{\mu(\mathbf{M}(A_r, B_r))}{\mu(B_r)} \leq \frac{2^d f(o)}{f(b)} \operatorname{Jac}(\mathbf{M_a})(b).$$

Remark 2. If $\operatorname{Jac}(\mathbf{M}_a)(b) \neq 0$ the existence of $\mathbf{I}$, U and V with the required assumptions is a consequence of the implicit function theorem. The case $\operatorname{Jac}(\mathbf{M}_a) = 0$ is quite special because then we have $\lim_{r \to 0^+} \frac{\mathcal{L}^d(\mathbf{M}(A_r, B_r))}{\mathcal{L}^d(B_r)} = 0$.

Proof. We give the proof for the Lebesgue measure $\mathcal{L}_d$. The second result is just a consequence of the first one and the continuity of f. It appears actually in the proof that $\mathbf{M}(A_r, B_r)$ (that contains o) has a diameter tending to 0 when r goes to 0. We take the points a, b, o, the sets B_r, A_r and the maps $\mathbf{I}, \mathbf{M}$ as in the statement of the theorem. We consider the set $M_r := \mathbf{M}(A_r, B_r)$ and will prove that it is included in a certain set whose $\mathcal{L}_d$-measure is equivalent to $\operatorname{Jac}(\mathbf{M}_a)\mathcal{L}_d(B_r)$. The key to this fact is

$$(6) \qquad \mathbf{M}(A_r, B_r) = \bigcup_{p,q \in B_r} \mathbf{M}(\mathbf{I}(p), q) = \bigcup_{p,q \in B_r} \mathbf{M}(\mathbf{I}(p), p + (q - p)).$$

The set $\mathbf{M}(A_r, B_r)$ shall have a small measure because each point $\mathbf{M}(\mathbf{I}(p), p+(q-p))$ is close to $o = \mathbf{M}(\mathbf{I}(p), p)$. We will use differentiation tools to quantify this idea. We assumed that $\mathbf{M}$ is C^∞-differentiable

on U. For any $q \in V$ let $\mathbf{M}_q$ be the map $\mathbf{M}(q, \cdot)$. We now write

$$
\begin{aligned}
(7) \quad \mathbf{M}(\mathbf{I}(p), & p + (q - p)) \\
= & 0 + D\mathbf{M}_{\mathbf{I}(p)}(p).(q - p) \\
& + \left[\mathbf{M}\left(\mathbf{I}(p), p + (q - p)\right) - D\mathbf{M}_{\mathbf{I}(p)}(p).(q - p) \right] \\
= & D\mathbf{M}_a(b).(q - p) + \left[\left(D\mathbf{M}_{\mathbf{I}(p)}(p) - D\mathbf{M}_a(b) \right).(q - p) \right] \\
& + \left[\mathbf{M}\left(\mathbf{I}(p), p + (q - p)\right) - D\mathbf{M}_{\mathbf{I}(p)}(p).(q - p) \right].
\end{aligned}
$$

For p and q close to b, the two last terms of the previous sum are small and can be bounded using the continuity of $D\mathbf{M}_{\mathbf{I}(p)}(p)$ and the second derivative of $\mathbf{M}$ at (a, b). When r tends to zero, p and q become close to b and

$$
\begin{aligned}
\sup_{p, q \in B_r} & \left| \left(D\mathbf{M}_{\mathbf{I}(p)}(p) - D\mathbf{M}_a(b) \right).(q - p) \right. \\
& \left. + \mathbf{M}(\mathbf{I}(p), p + (q - p)) - D\mathbf{M}_{\mathbf{I}(p)}(p).(q - p) \right| = o(r).
\end{aligned}
$$

Therefore, as $B_r - B_r = \{ v \in \mathbb{R}^d \mid v = p - q \quad p, q \in B_r \}$ is $\mathcal{B}(0, 2r)$, the Euclidean ball of center 0 and radius $2r$, the relations (6) and (7) give the following set inclusion

$$
(8) \qquad \mathbf{M}(A_r, B_r) \subset D\mathbf{M}_a(b).(\mathcal{B}(0, 2r)) + \mathcal{B}(0, \varepsilon(r)r)
$$

where $\varepsilon(r)$ is a non-negative function which tends to zero when r tends to zero. We observe now that the $1/r$-dilated set of the right-hand set in (8) is the $\varepsilon(r)$-parallel set of $D\mathbf{M}_a(b).(\mathcal{B}(0, 2))$. As the measure of $D\mathbf{M}_a(b).(\mathcal{B}(0, 2)) + \mathcal{B}(0, \varepsilon(r))$ tends to the one of $D\mathbf{M}_a(b).(\mathcal{B}(0, 2))$, the measure of $D\mathbf{M}_a(b).(\mathcal{B}(0, 2r)) + \mathcal{B}(0, \varepsilon(r)r)$ is equivalent to the measure of $D\mathbf{M}_a(b).(\mathcal{B}(0, 2r))$. But

$$
\mathcal{L}_d(D\mathbf{M}_a(b).(\mathcal{B}(0, 2r))) = 2^d \mathcal{L}_d(D\mathbf{M}_a(b).(\mathcal{B}(0, r))) = 2^d \operatorname{Jac}(\mathbf{M}_a)(b) \mathcal{L}_d(B_r)
$$

and we can conclude the claim by combining this identity with (8). $\quad\square$

We now give some examples of applications developed in [13] and [14].

Example 1. Consider the Heisenberg group H_d seen as $\mathbb{R}^d$ with the Carnot-Carathéodory distance ρ_{cc} and $\mathbf{M}$ the midpoint map $\mathcal{M}^{1/2}$ (in our previous papers we denoted H_d by $\mathbb{H}_n$ for $d = 2n+1$). Therefore for any x and y such that $\mathcal{M}^{1/2}$ is a single point $\{m\}$, we set $\mathbf{M}(x, y) = m$ and it is characterized by $\rho_{cc}(x, m) = \rho_{cc}(y, m) = \frac{1}{2}\rho_{cc}(x, y)$. Take $o = 0$, $b = (1, 0, \ldots, 0)$ and $a = (-1, 0, \ldots, 0)$. Then we proved in [14] that $\operatorname{Jac}(\mathbf{M}_a) = \frac{1}{2^{d+2}}$. In [14] we saw moreover that $\mathcal{L}_d(A_r) = \mathcal{L}_d(B_r)$. Therefore $BM_g(N)$ is not satisfied for any $N \geq 0$ because $\frac{1}{2}\mathcal{L}_d(A_r)^{1/N} + \frac{1}{2}\mathcal{L}_d(B_r)^{1/N} = \mathcal{L}_d(B_r)^{1/N}$ would imply $\mathcal{L}_d(\mathbf{M}(A_r, B_r))/\mathcal{L}_d(B_r) \geq 1^N$

which is eventually false for small enough r. Indeed with Theorem 1 we see that this ratio is asymptotically smaller than $\frac{1}{4} = \frac{2^d}{2^{d+2}}$.

Example 2. The Heisenberg group is not only a metric space, it is also a group. Prior to our work, there has been some research interest in BM_m for the Heisenberg group in relation with the isoperimetric problem [16, 21]. Leonardi and Monti proved for instance that $BM_m(3)$ holds in H_3. For N greater than d, we prove that $BM_m(N)$ does not hold in H_d by using Theorem 1. We keep the same points o, a, b as in Example 1 and $\mathbf{I}$ is now the map $p \to p^{-1}$. This transformation turns out to be simply the scalar multiplication by -1. Then A_r is a ball with the same measure as B_r. Moreover one can easily prove that $\mathrm{Jac}(\mathbf{M}_a) = 1$. Then $\mathcal{L}_d(\mathbf{M}(A_r, B_r))$ is asymptotically smaller than $2^d \mathcal{L}_d(B_r)$. Raise it to the power $1/N$ (for $N > d$) and it is smaller than $(\mathcal{L}_d(A_r))^{1/N} + (\mathcal{L}_d(B_r))^{1/N}$.

Example 3. The Grušin plane is the sub-Riemannian structure obtained on $\mathbb{R}^2$ when we take the "orthonormal basis" $(\partial_x, x\partial_y)$. Hence observe that outside the singular set $\{x = 0\}$, the two half-planes are non-complete Riemannian manifolds. In [13] we proved that $BM_g(K, N)$ does not hold for any K and N. Thanks to the dilations of this space, it is possible to reduce the argument to a contradiction of $BM_g(N)$. For that we consider as before the mid-point map for $\mathbf{M}$, we take $a = (0,0)$, $o = (-1, 0)$ and $b = (-2, 0)$. We have $\mathrm{Jac}(\mathbf{M}_a) = 2^{-4}$ and $\mathbf{I}$ is a local diffeomorphism near b with $\mathrm{Jac}(\mathbf{I}) = \frac{1}{7}$. Then $\mathcal{L}_2(A_r) \sim \frac{\mathcal{L}_2(B_r)}{7}$ and $\mathcal{L}_2(\mathbf{M}(A_r, B_r)) \leq f(r)$ with $f(r) \sim \frac{\mathcal{L}_2(B_r)}{4}$. Then one can easily see that $BM_g(N)$ does not hold for any N.

In the end of these notes we would like to present other possible examples to which Theorem 1 could apply. The theory of Lott–Villani and Sturm is recent and it is not clear which spaces satisfy $CD(K, N)$ and the implied property $BM_g(K, N)$. We have seen in Example 1 that the Heisenberg groups do not satisfy $BM_g(0, N)$. Nor do they satisfy $BM_g(K, N)$ for any K because as in Example 3 there is a scaling argument using dilations. Contact manifolds seem also to be excluded from the class of spaces satisfying $BM_g(K, N)$ because their nilpotent approximations (pointwise tangent limit in some Gromov–Hausdorff sense) are Heisenberg groups (possibly non-isotropic). As in Example 3 again it seems that for this question we can assume $K = 0$ because counterexamples could be found for a small enough scale. Indeed for small scales $\tau_s^{K,N}$ is almost equal to $s = \tau_s^{0,N}$. More generally it seems that there is no hope to find a space with some property $BM_g(K, N)$ in sub-Riemannian geometry. We sketch in two steps the rough idea

of what could be a proof of such a general result. Firstly compute $\mathrm{Jac}(\mathbf{M}_a)$ and $\mathrm{Jac}(\mathbf{I})$ for non-isotropic Heisenberg groups and Carnot-Carathéodory groups [10, 20] and disprove BM_g by using Theorem 1. Then write down a correct proof showing that the argument passes to the limit for the large class of sub-Riemannian structures that have these spaces as pointwise nilpotent approximations.

According to non-published computations of Bakry and his collaborators, the half Grušin plane (see Example 3), i.e. the Grušin plane intersected with $\{x > 0\}$, does not satisfy $CD(0, N)$ for any $N < +\infty$ when it is equipped with the weighted measure $\mu = xdxdy$. Note that this measure is different from $dxdy$ considered in Example 3 or from the Riemmanian volume, i.e. $\frac{dxdy}{|x|}$. This result can be proved by analyzing the Bakry - Émery criterion $CD_{BE}(0, N)$ because the half Grušin plane is simply a Riemannian manifold and μ is an invariant measure of an elliptic operator that does not satisfy the criterion. Surprisingly however $CD(0, \infty)$ (see [26] for a definition) seems to be satisfied, which gives evidence that changing the reference measure may provide an unexpected example. As far as we are concerned in this note, we would like to obtain another proof that $CD(0, N)$ is false for $N < +\infty$ by making use of Theorem 1.

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Institut Fourier BP 74, UMR 5582, Université Grenoble I, 38402 Saint-Martin-d'Hères Cedex, France
 E-mail address: nicolas.juillet@ujf-grenoble.fr

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN, POPPELS-DORFER ALLEE 82, 53115 BONN, GERMANY
E-mail address: juillet@wiener.iam.uni-bonn.de