New hexachordal theorems in metric spaces with a probability measure

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Abstract

The Hexachordal Theorem is a fancy combinatorial property of the sets in $\mathbf{Z}/12\mathbf{Z}$ discovered and popularized by the musicologist Milton Babbitt (1916-2011). Its has been given several explanations and partial generalizations. Here we complete the comprehension of the phenomenon giving both a geometrical and a probabilistic perspective.

1 An introductive example

For describing a set A, one can adopt a statistical method and look at the mean distance between two points picked randomly from A. To fix ideas, assume that A is a subset of the sphere $\mathbf{S}^2$ equipped with the chord distance d and the surface measure μ. By mean distance we mean $M_1(A) = \mu(A)^{-2} \iint_{A\times A} d(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)$ the value corresponding to $p = 1$ in the range of the power mean distances $(M_p(A))_{p>0}$ where

$$M_p(A) := \left(\frac{1}{\mu(A)^2} \iint_{A\times A} d(x,y)^p\, \mathrm{d}\mu(x)d\mu(y)\right)^{1/p}. \tag{1}$$

It is clear that rotating A on $\mathbf{S}^2$ does not modify $M_1(A)$. To state the obvious the other power mean distances –as for instance the quadratic mean distance $M_2(A)$– are also conserved after rotation. Finally the (essential) diameter of A is obviously conserved – besides the fact it is $\lim_{p\to\infty} M_p(A) = \sup_p M_p(A)$. Nothing surprising in all that: the set A is "the same" before and after rotation.

As we will prove in this paper, if $\mu(A) = \mu(\mathbf{S}^2)/2$ another –this time nontrivial– operation conserves $M_1(A)$ and any other power mean $M_p(A)$, namely the complementary map:

$$A \mapsto A^c := \mathbf{S}^2 \setminus A.$$

Not only are the power means conserved but also the complete *distribution* –or *law*, a central notion from probability theory we recall later– of the random distance between two independent points[1]. To give a concrete and, we think, surprising example one can consider A to be the set of points with latitude between -30° and 30°, as illustrated in Figure 1. As we just said we have $M_p(A) = M_p(A^c)$ for every

[1]Since the diameter of $\mathbf{S}^2$ is finite, we face the classical *Hausdorff moment problem* in which the distribution of the random distance is uniquely characterized by $(M_p(A))_{p\in\mathbb{N}}$. However, we prove the invariance independently of this argument and for non bounded spaces as well.

$p \geq 0$. But having the same distribution implies other identities: for instance the probability that the random distance is smaller than $\sqrt{2}$ (the distance between the poles and the equator) is the same for A and A^c.[2]

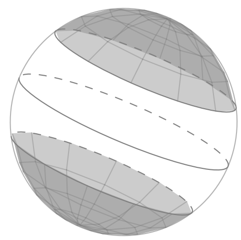

Figure 1: Two points randomly picked in the bright region of the sphere have distance distributed equally as the one between points picked in the dark region (made of two caps).

This type of singular invariance phenomenon may have been discovered a couple of times. We are in particular aware of two special instances in science and arts, namely in crystallography ([18, 8, 19]) and in music, where it was made famous by the American composer and music theorist Milton Babbitt under the name *Hexachordal Theorem* [5]. More precisely the phenomenon described above for $\mathbf{S}^2$ was proved for the continuous circle $\mathbf{S}^1$ (in a generalized version [6, 15, 16]) and the discrete circle $\mathbf{Z}/12\mathbf{Z}$ in Babbitt's original version. Cyclic groups are in fact traditionally used to represent musical structures such as chords, melodies or rhythms. In particular, a hexachord is a subset of 6 notes over the 12 of the chromatic scale $\mathbf{Z}/12\mathbf{Z} \equiv \{C, C\#, D, \ldots, C\}$. Babbitt realized that the same intervals appear with the same multiplicity in the hexachord A^c as in the hexachord A.

In this paper we give a full characterization of the spaces that host such a hexachordal phenomenon, see Theorem 4.2 and Theorem 4.6. However, for a progressive introduction in the matter, we start in Theorem 1.3 with a sufficient condition on the space with one implication only, see Theorem 1.3 called *common growth condition* (or *(CGC)* for short) and its application to Babbitt's case $\mathbf{Z}/12\mathbf{Z}$ just after.

1.1 A first glance at the proof for the sphere $\mathbf{S}^2$

For the initial example the proof of the invariance is simple enough that we can already deliver the bulk of its substance now. Let $(x_1, y_1), \ldots, (x_N, y_N)$ be a large sample of pairs of random points from $\mathbf{S}^2$. Intuition (backed up by theorems such as the law of large numbers) tells us that we can estimate $M_d(A)$ and $M_d(A^c)$ by collecting pairs (x_k, y_k) such that $x_k \in A$ and $y_k \in A$ on one side and pairs satisfying $(x_k, y_k) \in (A^c)^2$ on the other side. Note that the expected size of these subsamples is $N/4$. We claimed that the distribution of the distance is

[2] With simple geometric considerations one proves that this probability is 1/2.

conserved by $A \mapsto A^c$ so our two samples $\{d(x_k, y_k) : (x_k, y_k) \in A^2 \}$ and $\{d(x_k, y_k) : (x_k, y_k) \in (A^c)^2 \}$ should have the same statistical aspect.

The deep reason for this observation is only revealed once a third sample is concatenated to the two others, namely $\{d(x_k, y_k) : (x_k, y_k) \in A \times A^c\}$. Doing this, the first sample becomes the distances $d(x_k, y_k)$ with $x_k \in A$ and $y_k \in A \cup A^c = \mathbf{S}^2$, i.e., no restriction on y_k, whereas the second is made of the pairs $(x_k, y_k) \in \mathbf{S}^2 \times A^c$—again one point, here x_k, is free. Now it appears that in both cases we are considering the typical random distance to one given point of the sphere. The fact that this point, x_k (respectively y_k) is in A (respectively A^c) has no incidence on the random distance[3]. Therefore, the two augmented samples have the same properties (up to variations due to the sampling) and since we added the same sample to both, so do the initial samples.

1.2 Statement of the generalized hexachordal theorem

Recall that the distribution of a $\mathcal{M}$-valued random variable Z is the probability measure on the measured space $(\mathcal{M}, \mathfrak{M})$ that is denoted by $\mathbf{P}(Z \in \cdot)$ and defined by $E \in \mathfrak{M} \subseteq \mathcal{P}(\mathcal{M}) \mapsto \mathbf{P}(Z \in E) = \mathbf{P}(\{\omega \in \Omega : Z(\omega) \in E\})$. It becomes a conditional distribution $\mathbf{P}(Z \in \cdot | \, C)$ if the probability measure $\mathbf{P}$ is biased by an event C that is assumed to be satisfied. The value $\mathbf{P}(Z \in E | \, C)$ is defined by $\mathbf{P}(C)^{-1}\mathbf{P}(\{Z \in E\} \cap C)$.

In our introductive example the probability space may be chosen to be $\Omega = \mathbf{S}^2 \times \mathbf{S}^2$ with $X(x, y) = x$, $Y(x, y) = y$ for every $(x, y) \in \mathbf{S}^2 \times \mathbf{S}^2$, and the probability measure $\mu(\mathbf{S}^2)^{-2}(\mu \times \mu)$. The random variable Z is $\mathbf{R}$-valued. It is $D = d(x, y)$. Finally, the event is $C = A \times A$ or $A^c \times A^c$.

Hereafter, $(\mathfrak{X}, d)$ is a separable metric space and μ a Borel σ-finite measure on it. We will refer of such triples $(\mathfrak{X}, d, \mu)$ as *metric measure spaces* and *metric probability spaces* if μ is a probability measure[4]. As suggested in the introduction the spaces we have in mind may be continuous spaces as well as discrete spaces.

We now introduce the common growth condition on $(\mathfrak{X}, d, \mu)$ which provides a sufficient condition for the main result of this paper.

Definition 1.1 (Common growth condition)**.** A metric measure space $(\mathfrak{X}, d, \mu)$ is said to satisfy the *common growth condition* if there exists a function ρ on $[0, \infty)$ such that for any center $x \in \mathfrak{X}$ and radius $r \in [0, \infty)$ the ball $\mathcal{B}(x, r) = \{y \in \mathfrak{X} : d(x, y) \leq r\}$ has measure $\rho(r)$. This also writes:

$$\forall x, y \in \mathfrak{X}, \forall r \geq 0, \mu(\mathcal{B}(x, r)) = \mu(\mathcal{B}(y, r)). \tag{CGC}$$

Remark 1.2. If a metric measure space $(\mathfrak{X}, d, \mu)$ satisfies the common growth condition for a function ρ, for every $x \in \mathfrak{X}$ this *common growth function* equals the *local growth function of center* x, i.e $\rho_x : r \mapsto \mu(\mathcal{B}(x, r))$. If μ is a finite measure we can introduce the *mean growth function* $\bar{\rho} := \mu(\mathfrak{X})^{-1} \int \rho_x \, \mathrm{d}\mu(x)$ so that the common growth condition is satisfied if and only if $\rho_x \equiv \bar{\rho}$ for every $x \in \mathfrak{X}$.

[3] This is a geometric property of $\mathbf{S}^2$ that will be catched by the common growth condition.

[4] Note that to any metric measure space of finite measure, we can associate a metric probability space $(\mathfrak{X}, d, \tilde{\mu})$ by normalization, i.e $\tilde{\mu} = \mu(\mathfrak{X})^{-1}\mu$.

We note that $\mathbf{S}^2$ and $\mathbf{Z}/12\mathbf{Z}$ satisfy the common growth condition since no point is different of the others. Formulated in a more classical mathematical way, their groups of isometries acts transitively on them. In Section 3 we present other examples of spaces with the common growth condition that are in the class of transitive examples as well as outside this class. Doing this we exhibit for the first time "non-transitive" spaces where the hexachordal property is satisfied.

We can now state our hexachordal theorem for metric probability spaces.

Theorem 1.3 (Hexachordal theorem for metric probability spaces)**.** *Let $(\mathfrak{X}, d, \mu)$ be a metric probability space. Assume that it satisfies the common growth condition. Then for every Borel set A of μ-measure $1/2$, with notation $A^c = \mathfrak{X} \setminus A$ it holds*

$$\mu^2 \left\{ (x, y) \in A^2 : \, d(x, y) \in E \right\} = \mu^2 \left\{ (x, y) \in (A^c)^2 : \, d(x, y) \in E \right\}. \qquad \text{(Hex)}$$

for every open subset $E \subset [0, \infty)$, where μ^2 is the product measure $\mu \times \mu$ used for the (measurable) sets of pairs $(x, y) \in \mathfrak{X}^2$.

Let us show how this theorem specializes to Babbitt's theorem. On the cyclic group $\mathbf{Z}/12\mathbf{Z}$ we consider the distance defined by

$$d(x, y) = \min_{k \in \mathbf{Z}} |x - y + 12k|.$$

Since this formula corresponds to the minimum number of steps ± 1 in $\mathbf{Z}/12\mathbf{Z}$ necessary to move from x to y, the distance d is the classical graph distance, the edges being distributed here exactly between the consecutive numbers of $\mathbf{Z}/12\mathbf{Z}$. The graphs appearing further in the paper are also endowed with their own graph distance. By choosing for μ the normalized counting measure on $\mathbf{Z}/12\mathbf{Z}$, i.e. $\mu(A) = \#A/12$ we obtain the following expression for (Hex) in Theorem 1.3:

$$\frac{1}{12^2} \#\{(x, y) \in A^2 : \; d(x, y) \in E\} = \frac{1}{12^2} \#\{(x, y) \in (A^c)^2 : \; d(x, y) \in E\}.$$

Babbitt's formulation is slightly different. Denoting by ψ_A and I_A the functions defined for $k \in \mathbf{N}$ by

$$\psi_A(k) = \# \left\{ (x, y) \subseteq A^2 : \; d(x, y) = k \right\}$$

and for $k \in \mathbf{Z}/12\mathbf{Z}$ by

$$I_A(k) = \# \left\{ (x, y) \subseteq A^2 : \; y - x = k \right\}$$

respectively counting the number of oriented pairs at distance $k \in \mathbf{N}$ and of oriented intervals $k \in \mathbf{Z}/12\mathbf{Z}$, (Hex) equivalently writes $\psi_A = \psi_{A^c}$. Next, for every A (and A^c) $I_A(k) = \psi_A(k)$ for $k = 0$ and $k = 6$ and, since $(x, y) \in A^2 \Leftrightarrow (y, x) \in A^2$, we also have $I_A(k) = I_A(12 - k) = \psi_A(k)/2$ for $k = 1, \ldots, 5$. Thus $I_A = I_{A^c}$ holds on the whole $\mathbf{Z}/12\mathbf{Z}$. The latter is Babbitt's formulation of his hexachordal theorem. The function I_A is classically called the *interval content* of A.

Hereafter we distinguish two types of hexachordal theorems: the metric ones as Theorem 1.3 and the general ones, presented in Section 4.2, where the distance fonction d is replaced by a general function f on $\mathfrak{X} \times \mathfrak{X}$. Examples include symmetric and antisymmetric functions, typically $f : (x, y) \mapsto x^{-1} \cdot y$ where $(\mathfrak{X}, \cdot)$ is a group. One recovers for the group $\mathbf{Z}/12\mathbf{Z}$ the interval content formulation $I_A = I_{A^c}$ by Babbitt. Since Babbitt's original formulation [5] and its first complete proof by Ralph Hartzler Fox [9], the Hexachordal Theorem has been discussed, reproved and sometimes generalized by several authors including David Lewin [13], Howard J. Wilcox [21], Steven K. Blau [7], Daniele Ghisi [10], Emmanuel Amiot [3] and Brian J. McCartin [11]. Among the metric theorems, one finds a full characterization of simple graphs exhibiting the Hexachordal property by T. A. Althuis and F. Göbel [2]. The Hexachordal property has also been studied by David Lewin in [14] within the framework of *Generalized Interval Systems*.

A concept related to our topic is the one of *homometric sets* with a meaning that may vary between the authors and the domain, see e.g. [17, 19, 1] and the literature therein. In our language two sets A and B are homometric if for every E

$$\mu^2 \left(\{(x, y) \in A^2 : d(x, y) \in E\} \right) = \mu^2 \left(\{(x, y) \in B^2 : d(x, y) \in E\} \right).$$

The hexachordal theorem simply states that A and A^c of measure $1/2$ are homometric.

2 Probabilistic interpretation and proof of Theorem 1.3

Our proof uses a probabilistic writing of (Hex). Let (X, Y) be a pair of $\mathfrak{X}$-valued independent[5] random variables of law μ and $D = d(X, Y)$. Property (Hex) writes

$$\mathbf{P}(X \in A \text{ and } Y \in A \text{ and } D \in E) = \mathbf{P}(X \in A^c \text{ and } Y \in A^c \text{ and } D \in E). \tag{2}$$

Adding $\mathbf{P}(X \in A \text{ and } Y \in A^c \text{ and } D \in E)$ on both sides we see that (Hex) holds if (and only if) one has

$$\mathbf{P}(X \in A \text{ and } D \in E) = \mathbf{P}(Y \in A^c \text{ and } D \in E) \tag{3}$$

for every Borel set $E \subseteq \mathbf{R}$. Hence, for Theorem 1.3 it suffices to prove (3).

[5]We recall to the readers that the independence of X and Y means the equation $\mathbf{P}(X \in S \text{ and } Y \in T) = \mu(S)\mu(T)$ holds for all measurable sets S and T of $\mathfrak{X}$.

Proof of Theorem 1.3. Let S be a Borel set of $\mathfrak{X}$ and $r \geq 0$. We have:

$$\begin{aligned}
\mathbf{P}(X \in S \text{ and } D \in [0,r]) &= \iint_{\mathfrak{X}\times\mathfrak{X}} \mathbb{1}(x \in S) \cdot \mathbb{1}(d(x,y) \leq r) \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&= \int_S \left(\int_{\mathfrak{X}} \mathbb{1}(d(x,y) \leq r) d\mu(y) \right) \mathrm{d}\mu(x) \qquad (4) \\
&= \int_S \mu(\mathcal{B}(x,r))\mathrm{d}\mu(x) \\
&= \mu(S) \cdot \rho(r).
\end{aligned}$$

This proves that X and D are independent random variables, X has law μ (this is not new) and D has cumulative distribution function ρ (see Remark 2.3). Therefore, on the left-hand side of (3), $\mathbf{P}(X \in A \text{ and } D \in E) = \mathbf{P}(X \in A) \times \mathbf{P}(D \in E) = (1/2)\mathbf{P}(D \in E)$. Exactly in the same way (or noticing that (X, D) and (Y, D) have the same joint law) we see that Y and D are independent and $\mathbf{P}(Y \in A^c \text{ and } D \in E) = (1/2)\mathbf{P}(D \in E)$. This proves (3) and hence completes the proof. ☐

Remark 2.1. We can express (Hex) in a different way in terms of conditional laws. Dividing Equation (2) by $\frac{1}{4} = \mathbf{P}((X,Y) \in A^2) = \mathbf{P}((X,Y) \in (A^c)^2))$ we obtain

$$\mathbf{P}(D \in \cdot |\, X \in A \text{ and } Y \in A) = \mathbf{P}(D \in \cdot |\, X \in A^c \text{ and } Y \in A^c).$$

This may be read as follows: Provided points X and Y are in A, their distance D is distributed in the same way as it were provided they were in the complementary set.

Remark 2.2. Similarly, $\mathbf{P}(D \in \cdot |\, X \in A) = \mathbf{P}(D \in \cdot |\, Y \in A^c)$ is a version of (3) formulated with conditional laws. The following one-line computation

$$\mathbf{P}\left(D \leq r |\, X \in A\right) = \mu(A)^{-1} \int_A \underbrace{\mathbf{P}(d(x,Y) \leq r)}_{=\rho_x(r)=\rho(x)} d\mu(x) = \rho(r),$$

with its counterpart $\mathbf{P}\left(D \leq r |\, Y \in A^c\right) = \rho(r)$ (for every $r \geq 0$), constitute an alternative, shorter and more probabilistic proof of Theorem 1.3.

Remark 2.3. Taking $S = \mathfrak{X}$ in (4) for a general $\mathfrak{X}$ we obtain $\mathbf{P}(D \leq r) = \bar{\rho}(r)$ so that $\bar{\rho}$ (introduced in Remark 1.2 for general metric measure spaces) is the cumulative distribution function of D. The cumulative distribution functions of $d(x,Y)$ and $d(X,y)$ are simply ρ_x and ρ_y. Moreover, under the common growth condition all these functions equal ρ.

Remark 2.4. The random variables X, Y and D are pairwise independent but they are not independent. In particular, for very localized sets S and T, say balls of (small) radius ε, the law $\mathbf{P}(D \in \cdot |\, X \in S \text{ and } Y \in T)$ is a measure concentrated on an interval of length 2ε, hence different from $\mathbf{P}(D \in \cdot)$.

3 Metric probability spaces satisfying the common growth condition

We give examples of spaces where Theorem 1.3 applies.

3.1 Transitive examples

A classical group-theoretical framework which ensures (CGC) is the one of transitive group actions. We assume that for every x and y in $\mathfrak{X}$ there exists a map $f : \mathfrak{X} \to \mathfrak{X}$ that satisfies

- $f(x) = y$,
- $d(f(z), f(z')) = d(z, z')$ for every $z, z' \in \mathfrak{X}$, so that f is an isometry,
- $f_{\#}\mu = \mu$, where $f_{\#}\mu := \mu(f^{-1}(\cdot))$ is the law of $f(X)$ if X has law μ.

The common growth condition follows:

$$\rho_x(r) = \mu(\mathcal{B}(x, r)) = \mu(f^{-1}\mathcal{B}(f(x), r)) = \mu(\mathcal{B}(f(x), r)) = \rho_y(r).$$

Let us first focus on the class of transitive examples satisfying (CGC) in the discrete setting of finite graphs with their counting measure. Note that in this case the condition $f_{\#}\mu = \mu$ is automatically satisfied because f is one-to-one. Graphs satisfying the two first conditions are usually called (vertex-)transitive graphs and f a graph isomorphism. Many of these graphs are the Cayley graph of a finite group. Recall that if a group $\mathfrak{X}$ is generated by a finite system of generators $\Sigma \subseteq \mathfrak{X}$, the Cayley graph attached to $(\mathfrak{X}, \Sigma)$ is the graph with vertices $\mathfrak{X}$ and edges the pairs (x, y) such that $x^{-1}y \in \Sigma$ or $y^{-1}x \in \Sigma$. As usual we denote this adjacency relation by $x \sim y$. Let us check that these spaces $\mathfrak{X}$ with the counting measure and the path distance are of transitive type and hence satisfies the common growth condition. Given x and y we choose for f the translation defined by $\tau_v : z \mapsto vz$ where $v = yx^{-1}$, so that $f(x) = y$. It is an isometry because $\tau_v(z) \sim \tau_v(z')$ if and only if $z \sim z'$, which follows from $\tau_v(z)^{-1}(\tau_v(z')) = z^{-1}z'$. Finally recall that it preserves the counting measure since it is one-to-one. Basic examples of this type (i.e finite Cayley graphs) are:

- The symmetric group $\mathfrak{S}(n)$ with, for instance, for Σ the set of transpositions.
- The group $(\mathbf{Z}/n_1\mathbf{Z}) \times \cdots \times (\mathbf{Z}/n_k\mathbf{Z})$ with $\Sigma = \{(0, \ldots 0, \pm 1, 0, \ldots, 0)\}$. Note that for $n_1 = \cdots = n_k = 2$ we find the so-called hypercube $\{0, 1\}^k$. For $k = 1$ and $n_1 = 12$ we recover the chromatic scale $\mathbf{Z}/12\mathbf{Z}$.

Note that a vertex-transitive graph may not be the Cayley graph attached to some $(\mathfrak{X}, \Sigma)$, a counterexample being the *Petersen graph* (a famous graph with 10 vertices and 15 edges) another one the graphs made of the edges and vertices of the *dodecahedron*, *icosahedron* and the *truncated icosahedron*

We list now some continuous transitive examples among the most basic:

- The (hyper)torus $\mathbf{T}^d = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ of dimension d with its normalized volume (among other tori).
- Any sphere $\mathbf{S}^d$ (see Remark 3.1) or product of spheres with their normalized volumes.
- The Klein bottle. When the Euclidean space $\mathbf{R}^2$ is made a quotient through the group spanned by the translation $(x, y) \mapsto (x + 1, y)$ and the glide reflexion $(x, y) \mapsto (1 - x, y + 1)$[6] the translations of $\mathbf{R}^2$ remains isometries that are acting transitively. Topologically the quotient space a Klein bottle with fundamental domain the square $[0, 1) \times [0, 1)$ (the lower and and upper sides are identified after inversion of the orientation). Any Klein bottle of volume 1 obtained in a similar way will be transitive, satisfy the common growth condition and hence host the hexachordal property. Note that two such Klein bottles are generally not isometric.
- For more exotic examples we can think to Albanese tori. Their topology is different from the one of usual tori.

Remark 3.1. Based on their algebraic structures (of sets of complex numbers, quaternions or octonions of modulus 1, respectively) the spheres $\mathbf{S}^1$, $\mathbf{S}^3$ and $\mathbf{S}^7$ were already considered in [15, §7] for algebraic generalizations of the Hexachordal Theorem (in the spirit of Examples 4.7 and 3.6), the case of spheres of other dimension remaining open. For general spheres a proof similar to ours is briefly suggested in the Open Problems of [6] after it is completed for $\mathbf{S}^1$. However, it does not seem directly implementable on the Patterson functions (see Remark 4.15 for this notion).

Remark 3.2. The Hexachordal Theorem has been largely studied in the so-called *Transformational Music Theory* of Lewin (see [14]) in particular in the context of (transformation) groups T acting on a musical space S in a simply transitive way. The uniquely determined group element mapping x to y is called *interval* and denoted by $\mathrm{Int}(x, y)$. By choosing $e \in S$ as a reference we can identify T with S through $x \in S \mapsto \mathrm{Int}(e, x) \in T$. In this way the group action of $\mathrm{Int}(x, y)$ is identified with the left product by $z \mapsto (yx^{-1})z$. The triplet (S, T, Int) was called *generalized interval system* or *GIS* by David Lewin. This is the language for the proof of the hexachordal theorem for locally compact groups obtained in [15, 16].

3.2 Non transitive examples

The metric measure spaces of this subsection are particularly interesting since they are of different nature from the previous ones. These are graphs that satisfy (CGC) - and hence (Hex) - but are not transitive (we call them *non transitive*). Example 3.4 is with seven vertices the smallest possible non transitive simple graph. During the writing of the present paper we realized that a collection of similar graphs (notably three graphs with twelve vertices) were already exhibited by Althuis

[6]The elements of this group are the isometries of the form $(x, y) \mapsto (k \pm x, y + l)$ where $\pm$ is $+$ if and only if l is even.

and Göbel in [2]. The phenomenon of non transitive spaces satisfying (CGC) is not limited to graphs: other examples can be obtained by taking the product of such graphs with e.g. $\mathbf{S}^1$, see Examples 3.6. Example 3.9 shows an alternative construction preserving (CGC). The question whether non transitive spaces with (CGC) can be found among Riemannian manifolds appears very interesting to us. Next subsection gives a negative answer in the case of surfaces.

Example 3.3. Consider the finite 3-regular graph depicted on Figure 2. One can easily convince that it satisfies the common growth condition: the balls of radius 0 have cardinal 1, the balls of radius 1 cardinal 4 and all the greater balls are the whole space whose cardinal is 8. However, it clear that a and h are points of different types: the neighbors of h are disconnected whereas the neighbors b and c of a satisfy $b \sim c$. Consequently the group of isomorphisms does not act transitively.

For the sake of completeness we provide the distribution of $d(X,Y)$ for the part $A = \{a, b, c, d\}$. The reader can check that it is the same as the one attached to A^c.

r	0	1	2
$\mathbf{P}(D = r \mid (X,Y) \in A^2)$	$4/16$	$8/16$	$4/16$

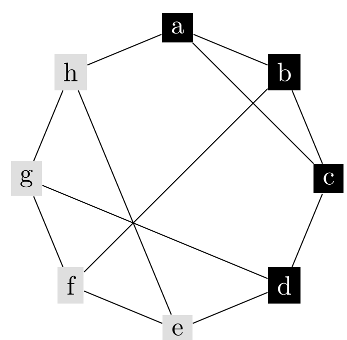

Figure 2: Two points randomly picked in the dark region of the graph have distance equally distributed as the one between points picked in the bright region.

Example 3.4. The graph of Figure 3 also satisfies the common growth condition ($\rho(0) = 1$, $\rho(1) = 5$, $\rho(2) = 7$). With cardinal seven it has the minimal cardinal for a graph satisfying (CGC) without transitive action of group of isomorphisms. However, since seven is an odd number the hexachordal property is –contrary to Example 3.3– a trivial statement: subsets A and A^c of cardinal $7/2$ do not exist. Hence Theorem 1.3 is a correct but empty statement.

Theorem 4.2 in the next section will give a new turn to this poor conclusion. Let us already give an idea of it: if we split point a in two parts and join it for half to $A_0 = \{b, b, c, c, d, d\}$ and for half to $A_1 = \{e, e, f, f, g, g\}$, two independent random points in A_0 (augmented with the half point) will have distance D_0 distributed in the same way as the random distance D_1 between two points in the complement.

The reader can check that both times it is the distribution given in the following tabular.

r	0	1	2
$\mathbf{P}(D_i = r)$	$13/49$	$32/49$	$4/49$

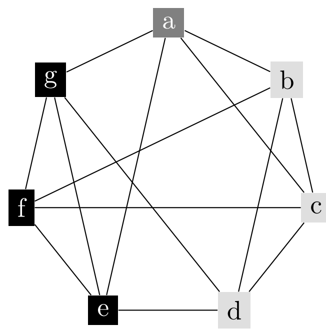

Figure 3: Two points randomly picked in the dark region of the graph have distance distributed equally as the one between points picked in the bright region. Vertex a is for half bright and for half dark.

Remark 3.5. The graph of a star drawn in a heptagon compared with Example 3.4 provides the evidence that two different (non isomorphic) spaces can satisfy the common growth condition with exactly the same common growth function[7]. Another famous example[8] has been exhibited by Kowalski and Preiss in [12]: the Euclidean space $\mathbf{R}^3 \subseteq \mathbf{R}^4$ and the cone $C = \{x \in \mathbf{R}^4 : \ x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ with, for both spaces, the (Euclidean induced by $\mathbf{R}^4$) chord distance and the induced volume (of dimension 3) satisfy (CGC) with the same common growth function. Notice that $\mathbf{R}^3$ is transitive and (C, d, μ) is not: It possesses for instance a singularity at 0. A uniqueness result for C among subsets of Euclidean spaces is established in [12].

3.3 The case of Riemannian manifolds of dimension 2

We started the paper with a surface, namely the sphere $\mathbf{S}^2$ that satisfies the common growth condition with the chord distance and its surface measure (normalized to become a probability measure). Notice now that the geodesic distance (for which the distance between two points is the length of the shortest path drawn on the surface) can be expressed as an increasing function of the chord distance, hence the common growth condition also holds for the geodesic distance. In the following we enlarge the exploration and look at all the surfaces with their geodesic distance that we classify with respect to the common growth condition. Precisely we consider the Riemannian manifolds $\mathfrak{X}$ of dimension 2 that we moreover assume to

[7]As a corollary these spaces are homometric.
[8]but unfortunately for measures of infinite mass

be connected, complete and separable. We consider them with their geodesic distance d and the corresponding Riemannian volume μ. Since we work with metric probability spaces we incidentally assume $\mu(\mathfrak{X}) = 1$.

Assume now that the surface $\mathfrak{X}$ satisfies the common growth condition. The first consequence of it is that the (sectional) curvature is constant. Namely it is well established that at any point $x \in \mathfrak{X}$,

$$\mu(\mathcal{B}(x,r)) =_{r\to 0^+} \pi r^2(1 - \kappa(x)r/24) + o(r^3)$$

where $\kappa(x)$ is the curvature at x. It follows that κ can be expressed independently of x since it is $\kappa(x) = \lim_{r\to 0} 24(\pi r^2 - \rho(r))/r$. Therefore we enter the well-known class of surfaces of constant curvature. Up to multiplying the distance by a positive constant, such a Riemannian manifold is known to have for universal cover one of the three simply connected "space forms": the Euclidean space (of curvature zero), the hyperbolic plane (curvature -1) and the sphere (curvature 1). Therefore, up to a scaling that makes it satisfy $\mu(\mathfrak{X}) = 1$, our Riemannian manifold $\mathfrak{X}$ is a quotient of one of these three spaces. In particular $\mathfrak{X}$ is locally isometric to one of these spaces so that $\rho_x(r) = \rho_{x'}(r)$ for any $x, x' \in \mathfrak{X}$ and r close enough to zero.

- For curvature zero we obtain the 2-tori and the Klein bottles. The former are obtained as the quotient of $\mathbf{R}^2$ through $\mathbf{Z}u+\mathbf{Z}v$ where $|\det(u,v)|$ is equal to 1, since it is the volume of a fundamental domain, i.e. the parallelogram $\{\alpha u + \beta v \in \mathbf{R}^2 : \alpha, \beta \in [0,1[\}$. Note that two 2-tori are generally not isometric to each other. The same discussion is valid for Klein bottles (recall also p. 8).

- For the positive curvature we obtain only two examples: the sphere[9] of radius $1/\sqrt{4\pi}$ and the projective two plane $\mathbf{RP}^2$ obtained from the sphere of radius $1/\sqrt{2\pi}$ when the opposite points are identified.

- For the negative curvature we dive in the rich world of hyperbolic surfaces where we show that no surface satisfies the common growth condition. As for the other space forms mentioned above, any two points $x, x' \in \mathfrak{X}$ possess isometric neighborhoods and $\rho_x(r) = \rho_{x'}(r)$ for r small enough. However when the radii of $\mathcal{B}(x,r)$ and $\mathcal{B}(x',r)$ increase, some balls will overlap themselves earlier than others. This happens at the so-called *cut locus*. Recall that hyperbolic surfaces of volume one may be compact or not. Briefly, in the compact case we take x on the so-called systol and x' not on it. For non-compact hyperbolic surfaces of volume one we see that outside large balls (of radius R and center x_0) there are still balls of radius 1 but their volume uniformly tends to zero as R goes to infinity. Therefore, the common growth condition can not be satisfied.

[9] Up to the scaling we recover our introductive example $\mathbf{S}^2$ (for which μ was the Riemannian volume divided by 4π).

3.4 Constructions with metric spaces satisfying the common growth condition

Example 3.6 (Products). Let $(\mathfrak{X}_1, d_1, \mu_1)$ and $(\mathfrak{X}_2, d_2, \mu_2)$ be two spaces satisfying (CGC) for ρ_1 and ρ_2, respectively. Then the product space $\mathfrak{X} := \mathfrak{X}_1 \times \mathfrak{X}_2$ also satisfies it, with the product measure $\mu := \mu_1 \times \mu_2$. Several choices are possible to combine the distances. We focus on the ℓ^p norms of $(d_1, d_2) \in \mathbf{R}^2$, where $p \in [1, \infty]$:

- ℓ^∞ We can set $d((x_1, y_1), (x_2, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$. Let us comment on the example of the product of two finite graphs with their path distances. The resulting space is $\mathfrak{X}_1 \times \mathfrak{X}_2$ with the path distance resulting of the so-called *strong product* of the two graphs. In fact $d((x_1, y_1), (x_2, y_2)) \leq 1$ if and only if $d((x_1, y_1) \leq 1$ or $d((x_2, y_2) \leq 1$. Denoting by $(x_1, y_1) \simeq (x_2, y_2)$ the relation $\{(x_1, y_1) \sim (x_2, y_2)$ or $(x_1, y_1) = (x_2, y_2)\}$, it follows that $(x_1, y_1) \simeq (x_2, y_2)$ if and only if $x_1 \simeq x_2$ and $y_1 \simeq y_2$. One can check that (CGC) is satisfied for $\rho = \rho_1 \times \rho_2$.

- ℓ^p We can set $d^p((x_1, y_1), (x_2, y_2)) = d_1(x_1, y_1)^p + d_2(x_2, y_2)^p$ and obtain for this choice the common growth function $\rho(r) = \int_{[0,r]} \rho_2((r^p - t^p)^{1/p}) d\rho_1(t)$.

- ℓ^1 The product of two graphs is the so-called *Cartesian product* for which $(x_1, y_1) \sim (x_2, y_2)$ if and only if $(x_1 = x_2$ and $y_1 \sim y_2)$ or $(y_1 = y_2$ and $x_1 \sim x_2)$.

- ℓ^2 If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are isometrically embedded in Euclidean spaces, so is the product with the ℓ^2 distance. For instance the hexachordal phenomenon can be observed on $\mathbf{S}^1 \times \{0, 1\} \subseteq \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$.

Example 3.7 (Union of two spaces with the same common growth function). We consider for $i = 1, 2$ two spaces $(\mathfrak{X}, d_i, \mu_i)$ satisfying the common growth condition for the same function ρ. We assume moreover that the two spaces are bounded. As noticed in Remark 3.5 they can be different. Define by $\mathfrak{X}$ the disjoint union $\mathfrak{X}_1 \sqcup \mathfrak{X}_2$ with probability measure $\mu = (1/2)(\mu_1 + \mu_2)$ and distance defined by

$$d(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in \mathfrak{X}_i \text{ for some } i, \\ L & \text{otherwise.} \end{cases}$$

The common growth condition and (Hex) are satisfied for any $L \geq 0$ but in order to save the triangle inequality we have to require that for every $i = 1, 2$ the distance between any two points of $\mathfrak{X}_i$ is less than $2L$, i.e $L \geq \max(\mathrm{Diam}(\mathfrak{X}_1), \mathrm{Diam}(\mathfrak{X}_2))/2$. Theorem 4.6 treats of the symmetric functions f (or d) that may not be symmetric.

Example 3.8 (Graphs whose points are replaces by metric spaces). Let (G, d_0) be a finite graph with the common growth condition for the counting measure. We scale it so that adjacent points have distance L. We replace $G = \cup_{i=1}^N \{x_1\}$ by $\mathfrak{X} = \sqcup_{i=1}^N \mathfrak{X}_i$ a family of metric spaces $(\mathfrak{X}_i, d_i, \mu_i)$ with diameter smaller than $2L$ and satisfying (CGC) with moreover the same common growth function, $\rho_1 = \cdots = \rho_N$. On $\mathfrak{X}$ we set $d(x, y) = d_i(x, y)$ if $x, y \in \mathfrak{X}_i$ and $d(x, y) = d_0(x_i, x_j)$ if $x \in \mathfrak{X}_i$,

$y \in \mathfrak{X}_j$, $i \neq j$. It can be checked that the resulting space satisfies the common growth condition. A special case is Example 3.7.

Example 3.9. We have indicated examples of metric probability spaces $(\mathfrak{X}, d, \mu)$ satisfying (CGC) such that D is an absolutely continuous or a discrete random variable. With $\{0,1\} \times \mathbf{S}^1$ in Example 3.6 and the Examples 3.7 and 3.8 we see the possibility for D to have both a non trivial atomic and absolutely continuous part. We introduce now the situation of a space with the (CGC) such that D is diffuse but not absolutely continuous. Precisely its law is the Cantor law and its cumulative distribution function ρ is the Devil's staircase.

For $\mathfrak{X}$ we take the sequences $a = (a_1, a_2, \ldots)$ with $a_i \in \{0,1\}$ for every $i \geq 1$. The measure is the one of head/tail model, i.e we weight each digit with $1/2$ independently. For the the distance between a and b we set $d(a,b) = \sum_{i=1}^{\infty} |b_i - a_i|(2/3^i)$. Note that it corresponds to a ℓ^1 distance on an infinite product $\{0,1\}^{\mathbf{N}^*}$ weighted by a scaling factor $(2/3^i)$ on the i-th coordinate.

For an interesting application of Theorem 1.3 one can consider for A the set of sequences such that 0 follows the first 1 in the sequence. A simple exercise is to prove $\mu(A) = \mu(A^c) = 1/2$. A more challenging one is to prove that A and A^c are not isometric –even though they are homometric according to (Hex).

4 Full characterization of the spaces satisfying the hexachordal property

In this last section we show that (CGC) is not far from being a necessary and sufficient condition for the hexagonal property (Hex). To obtain this equivalence we i) observe that sets of measure zero have no incidence in the hexachordal property and introduce for this (CGC'), ii) carefully avoid the logical trap explained in Example 3.4 by introducing (Hex'). This being done we obtain Theorem 4.2. In Theorem 4.6 we give a second generalization that connects our work with previous group theoretic [21, 15] or abstract [9] interpretations of the hexachordal theorem.

4.1 Full characterization for metric probability spaces

For our full characterizations of Theorems 4.2 and 4.6 we introduce the concept of *balanced decomposition*. It is an appropriate answer to the problem described in Example 3.4. Similar concepts are to be found in the literature in the *weights* of [6] and the bounded functions of [15].

Definition 4.1. Let $(\mathfrak{X}, \mathfrak{F}, \mu)$ be a probability space. We call balanced decomposition of μ any pair (μ_0, μ_1) of probability measures such that $2\mu = \mu_0 + \mu_1$. Note that μ_0 and μ_1 can be identified with functions of density smaller than or equal to 2.

We can now state our full characterization of spaces that satisfy (Hex'), i.e. (Hex) generalized as suggested in Example 3.4.

Theorem 4.2 (Characterization for metric probability spaces)**.** *Let* $(\mathfrak{X}, d, \mu)$ *be a metric probability space. The following properties are equivalent:*

(CGC') *There exists a set $\mathfrak{X}' \subseteq \mathfrak{X}$ of full measure for μ such that the common growth condition is satisfied on $(\mathfrak{X}', d, \mu)$.*

(Ind) *For any independent random variables X and Y of law μ and $D = d(X, Y)$, the random variables X, Y and D are pairwise independent.*

(Hex') *For every balanced decomposition (μ_0, μ_1) of μ and two random triples $(X_i, Y_i, D_i)_{i=0,1}$ where for every i, (X_i, Y_i) is a pair of independent random variables of law μ_i and $D_i = d(X_i, Y_i)$, we have the equality on distributions*

$$\mathbf{P}(D_0 \in \cdot) = \mathbf{P}(D_1 \in \cdot).$$

Remark 4.3. We recover Theorem 1.3 as follows: the common growth condition implies (CGC') (take $\mathfrak{X}' = \mathfrak{X}$ for example). Hence (Hex') is satisfied for any balanced decomposition, in particular for (μ_A, μ_{A^c}) where A has measure $1/2$ and μ_A is defined by $\mu_A = \mu(A)^{-1}\mu(A \cap \cdot)$. This directly corresponds to (Hex) in Theorem 1.3, up to a factor 4.

Remark 4.4. If X and Y are independent of law μ, since d is symmetric we have equality of laws $(X, D) = (X, d(X, Y)) \sim (Y, d(Y, X)) = (Y, D)$. Therefore, to satisfy (Ind) it suffices that X and D are independent.

Remark 4.5. Following Remark 1.2 we see that (CGC') is satisfied if and only if $\rho_x = \bar{\rho}$ for almost every $x \in \mathfrak{X}$ where we recall from Remark 2.3 that $\bar{\rho} = \int \rho_x \, \mathrm{d}\mu(x)$ is the cumulative distribution function of D.

Proof of Theorem 4.2. The beginning of the proof of Theorem 1.3 is the implication (CGC)⇒(Ind). The reader can check that it also readily constitutes a proof of (CGC')⇒(Ind) too. We use that $x \mapsto \mu(\mathcal{B}(x, r))$ is equal to $\bar{\rho}(r)$ in all points x apart from a set of empty measure. Let us now prove (Ind)⇒(CGC'). For every $r \geq 0$ we set $S_r^- = \{x \in \mathfrak{X} | \, \rho_x(r) < \bar{\rho}(r)\}$ and $S_r^+ = \{x \in \mathfrak{X} | \, \rho_x(r) > \bar{\rho}(r)\}$. Recall that $\bar{\rho}$ is the cumulative distribution function of D and ρ_x the one of $d(x, Y)$. Suppose by contradiction that $\mu(S_r^-) > 0$. Thus

$$\mathbf{P}(X \in S_r^- \text{ and } D \in [0, r]) = \iint \mathbb{1}(x \in S_r^-) \cdot \mathbb{1}(d(x, y) \leq r) \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

$$= \int_{S_r^-} \left(\int \mathbb{1}(d(x, y) \leq r) d\mu(y) \right) \mathrm{d}\mu(x) = \int_{S_r^-} \mu(\mathcal{B}(x, r)) \mathrm{d}\mu(x) < \mu(S_r^-) \cdot \bar{\rho}(r),$$

which shows that X and D are not independent, a contradiction. Therefore $\mu(S_r^-) = 0$ and similarly $\mu(S_r^+) = 0$. We deduce that $\bigcup_{r \geq 0, \, r \in \mathbf{Q}} (S_r^- \cup S_r^+)$ has μ-measure zero. If we denote $\mathfrak{X}'$ the complementary set we obtain $\bar{\rho}(r) = \rho_x(r)$ for every $x \in \mathfrak{X}'$ and $r \in \mathbf{Q}$. This extends to every $r \in \mathbf{R}_+$ because cumulative distribution functions are right-continuous. Hence (CGC') is satisfied.

We have proved (CGC')⇔(Ind) and will be ready after we prove (Ind)⇔(Hex'). We postpone this proof to Theorem 4.6 because considering that d is symmetric and measurable on $\mathfrak{X} \times \mathfrak{X}$ this theorem states a result that includes (Ind)⇔(Hex'). Its proof is also independent from Theorem 4.2. Notice that d is measurable because (according to our definition of metric probability spaces) $\mathfrak{X}$ is separable so that the product σ-algebra on $\mathfrak{X}^2$ is induced by the product topology. Hence continuous functions as d are measurable. □

4.2 Full characterization for general spaces and groups

As suggested in Example 3.7 it is possible to replace d by a real symmetric function f. The reader can check that (CGC'), can appropriately be adapted from Remark 4.5 to provide a necessary and sufficient condition for (Hex'). The sets $\mathcal{B}(x,r) = \{y \in \mathfrak{X} | \ f(x,y) \leq r\}$ are no longer balls but the functions ρ_x and $\bar{\rho}$ of Example 1.2 still make sense. They also continue to be the cumulative distribution functions of $f(x,Y)$ and $F = f(X,Y)$.

In this section we show that f does not need to be real valued and that it even may not be symmetric. This point of view is considered since Babbitt's original theorem and further in Lewin's formalism (recall Remark 3.2). Typically f is an antisymmetric function as in Example 4.9 where it may be seen as an interval function.

Theorem 4.6 (Characterization for abstract probability spaces)**.** *Let $(\mathfrak{X}, \mathcal{F}, \mu)$ be a probability space and f a measurable symmetric function into a measured space $(\mathcal{M}, \mathfrak{M})$. The following properties are equivalent:*

- *(Ind) For any independent random variables X and Y of law μ and $F = f(X,Y)$, the random variables X, Y and F are pairwise independent*[10]*.*
- *(Hex') For every balanced decomposition (μ_0, μ_1), considering the triples (X_0, Y_0, F_0) and (X_1, Y_1, F_1), where for $i = 0,1$ the pair (X_i, Y_i) is made of independent random variables of law μ_i and $F_i = f(X_i, Y_i)$, we have equality of both distributions, $\mathbf{P}(F_0 \in \cdot) = \mathbf{P}(F_1 \in \cdot)$ as measures on $\mathcal{M}$.*
- *(Hex") For any balanced decompositions (μ_0, μ_1) and (ν_0, ν_1) where for $i = 0,1$, X_i has law μ_i, Y_i has law ν_i and $F_i = f(X_i, Y_i)$, we have equality of both distributions $\mathbf{P}(F_0 \in \cdot) = \mathbf{P}(F_1 \in \cdot)$.*

Moreover if f is no longer supposed to be symmetric (Ind)⇔(Hex") still holds as well as (Hex")⇒(Hex').

Proof. To complete the proof of Theorem 4.2 we first establish in part 1. and 2. of this proof the two implications of (Ind)⇔(Hex') in the case where f is symmetric. For the remainder, notice already that (Hex")⇒(Hex') is obvious since (Hex") corresponds to a generalization of (Hex') where the constraint $\mu_i = \nu_i$ is relaxed. In part 3. we will finish with the equivalence (Ind)⇔(Hex") by briefly adapting the scheme drawn up in 1. and 2.

1. (Ind)⇒(Hex'). Let us fix some measurable $E \subseteq \mathcal{M}$ and (μ_0, μ_1) a balanced decomposition of μ. We first prove

$$\mathbf{P}(f(x,Y) \in E) = \mathbf{P}(F \in E) \tag{5}$$

for μ-a.e. $x \in \mathfrak{X}$. This follows from the fact that these two functions have the same

[10] As explained in Remark 4.4, when f is symmetric (Ind) is satisfied if and only if X and F are independent.

integral on the measurable sets S in $\mathfrak{X}$. We have indeed

$$\begin{cases} \displaystyle\int_S \mathbf{P}(f(x,Y) \in E) d\mu(x) = \mathbf{P}(X \in S,\ \underbrace{f(X,Y)}_{F} \in E) \\ \displaystyle\int_S \mathbf{P}(F \in E) d\mu(x) = \mathbf{P}(X \in S) \cdot \mathbf{P}(F \in E) \end{cases}$$

Equality follows from (Ind). Integrating (5) with respect to μ_0 (that is absolutely continuous with respect to μ) we obtain $B_E(\mu_0, \mu) = B_E(\mu, \mu)$ where B_E is the bilinear function defined by $B_E : (\alpha, \beta) \mapsto \iint \mathbb{1}(f(x,y) \in E) d\alpha(x) d\beta(y)$. Note now that $f(x,Y) = f(Y,x)$ and that these random variables have also the same law as $f(X,x)$. Therefore $\mathbf{P}(f(X,y) \in E) = \mathbf{P}(F \in E)$ for μ-a.e. $y \in \mathfrak{X}$. Similarly as before we deduce $B_E(\mu, \mu) = B_E(\mu, \mu_1)$. Finally, subtracting $B_E(\mu_0, \mu_1)$ on each extreme side of $B_E(\mu_0, 2\mu) = 2B_E(\mu, \mu) = B_E(2\mu, \mu_1)$ we get

$$B_E(\mu_0, \mu_0) = B_E(\mu_1, \mu_1) \quad \text{for every measurable } E \subseteq \mathcal{M}. \tag{6}$$

Translated with random variables it is exactly (Hex').

2. (Hex')$\Rightarrow$(Ind). For this implication, it is sufficient to prove

$$\mathbf{P}(X \in S \text{ and } F \in E) = \mathbf{P}(X \in S) \cdot \mathbf{P}(F \in E)$$

for every measurable $E \subseteq \mathcal{M}$ and $S \subseteq \mathfrak{X}$ with $\mu(S) \geq 1/2$. For sets S of probability less than $1/2$ the independence relation is obtained through the complementary set $\mathfrak{X} \setminus S$. We fix S and E. Let μ_0 be $\mu(S)^{-1}\mu(\cdot \cap S)$ such that $(\mu_0, 2\mu - \mu_0)$ is a balanced decomposition of μ. Starting back from (6), adding $B_E(\mu_0, \mu_1)$ we obtain back $B_E(\mu_0, \mu) = B_E(\mu, \mu_1) = B_E(\mu_1, \mu) = B_E(\mu, \mu)$ where we use the symmetry of f in the second equality and the bilinearity in the third one. In probabilistic terms we have obtained

$$\mu(S)^{-1}\mathbf{P}(X \in S \text{ and } F \in E) = \mathbf{P}(F \in E),$$

which is exactly the wanted equation, since $\mu(S) = \mathbf{P}(X \in S)$.

3. We follow part 1. and obtain that $x \mapsto \mathbf{P}(f(x,Y) \in E)$ and $y \mapsto \mathbf{P}(f(X,y) \in E)$ are almost surely constant of value $\mathbf{P}(F \in E)$ on $(\mathfrak{X}, \mu)$. It follows

$$B_E(\mu_0, \nu_0 + \nu_1) = 2B_E(\mu_0, \mu) = 2B_E(\mu, \nu_1) = B_E(\mu_0 + \mu_1, \nu_1)$$

for every balanced decompositions (μ_0, μ_1) and (ν_0, ν_1). Subtracting $B(\mu_0, \nu_1)$ we obtain $B_E(\mu_0, \nu_0) = B_E(\mu_1, \nu_1)$ which proves the first implication. For the second one, from $B_E(\mu_0, \nu_0) = B_E(\mu_1, \nu_1)$ we obtain back $B_E(\mu_0, \nu) = B_E(\mu, \nu_1)$ for every $\mu_0 \leq 2\mu$ and $\nu_1 \leq 2\mu$ (these inequalities correspond to the conditions that $(\mu_0, 2\mu - \mu_0)$ and $(2\mu - \nu_1, \nu_1)$ are balanced decompositions). Choosing $\mu_0 = \mu(S)^{-1}\mu(\cdot \cap S)$ and $\nu_1 = \mu$ we can reconnect with the proof in 2. □

Example 4.7. Let $(\mathfrak{X}, \cdot)$ be a separable topological group (all operations are continuous) with a uniquely determined left(-translation)- and right(-translation)-

invariant probability measure μ (a Haar measure)[11]. This includes the locally compact Hausdorff topological groups of [15]. For f we first consider the product. For every $x \in \mathfrak{X}$ the law of $f(x, Y)$ is μ so that X and $F = X \cdot Y$ are independent. In the same way Y and F are independent (even though $\mathfrak{X}$ is not abelian) so that Theorem 4.6 applies. For an application of (Hex) one can consider the symmetric group $\mathfrak{S}(n)$ with its renormalized counting measure and the alternating group $\mathfrak{A}(n)$ for A. The random product of two independently chosen even permutations is distributed in the same way as for odd permutations. The two are in fact uniformly distributed on $\mathfrak{A}(n)$.

Example 4.8. In the same setting as Example 4.7 one can take the operation $f(x, y) = x^{-1} \cdot y$ (the Haar measure is conserved by $x \mapsto x^{-1}$). This function was often seen as an interval function in the literature of the hexachordal theorem.

Example 4.9. Let $(\mathfrak{X}, \mathcal{F}, \mu)$ be a probability space and f a measurable function defined on $\mathfrak{X} \times \mathfrak{X}$ with values in a measurable space $(\mathcal{M}, \mathfrak{M})$. We assume that f is antisymmetric in the sense there exists a measurable involution i on $\mathcal{M}$ (i.e a function $i : \mathcal{M} \to \mathcal{M}$ such that $i \circ i(m) = m$, for every $m \in \mathcal{M}$) with $f(x, y) = i \circ f(y, x)$. Then (Ind) is satisfied by X, Y and $F = f(X, Y)$ as soon as X and F are independent. To see this we consider $\mathbf{P}(Y \in S \text{ and } F \in E)$. It equals

$$\begin{aligned}
\mathbf{P}(Y \in S \text{ and } f(Y, X) \in i(E)) &= \mathbf{P}(X \in S \text{ and } f(X, Y) \in i(E)) \\
&= \mathbf{P}(X \in S \text{ and } F \in i(E)) \\
&= \mathbf{P}(X \in S) \cdot \mathbf{P}(f(X, Y) \in i(E)) \\
&= \mathbf{P}(Y \in S) \cdot \mathbf{P}(F \in E).
\end{aligned}$$

This show again that we have (Ind) in Example 4.8 without using the argument that X^{-1} has law μ.

Remark 4.10. For symmetric functions f the implication (Ind)⇒(Hex") holds even without assuming that Y and F are independent. However for general functions it is not sufficient to assume that $F = f(X, Y)$ is independent from X. Let us illustrate this with $f(x, y) = y$. Under μ_0 for X_0 and Y_0 independent the law of F_0 is actually μ_0. Under μ_1 the law of $F_1 =$ is $\mu_1 \neq \mu_0$. Therefore, (Hex') and (Hex") are false.

Example 4.11. Let us show that in Theorem 4.6 the implication (Hex')⇒(Ind) is false. To see this let $\mathfrak{X}$ be the space $\{\star, \#, \S, \bullet\}$ of cardinal 4 and the values of a non symmetric function f be given in the following table where X is the uniform choice of a row, Y of a column and F is the intersection. Our example is the right table of Figure 4, the left and the middle being part of the explanation. Observe that the law of $F = f(X, Y)$ conditioned on the choice of a row (the value of X) is not constant. Hence (Ind) is not satisfied. Observe now that (Ind) is satisfied for Cayley tables, i.e tables for a group structure (see e.g. [9, 21] where this fact

[11] Note that it is in fact enough that there exists a left-invariant measure μ. If X and Y are independent of law μ and μ', respectively where μ' is right-invariant (as for instance $\mu_{-1} : E \mapsto \mu(E^{-1})$), one can check that $Y \cdot X : \Omega \to \mathfrak{X}$ is measurable, is both left- and right-invariant and has law μ and μ'. Therefore, $\mu = \mu'$ so that there exists a unique Haar measure and it is bi-invariant.

	$\star$	$\#$	$\S$	$\bullet$
$\star$	0	1	2	3
$\#$	1	2	3	0
$\S$	2	3	0	1
$\bullet$	3	0	1	2

	$\star$	$\#$	$\S$	$\bullet$
$\star$	0	1	2	3
$\#$	1	2	3	0
$\S$	3	0	1	2
$\bullet$	2	3	0	1

	$\star$	$\#$	$\S$	$\bullet$
$\star$	0	1	3	3
$\#$	1	2	3	0
$\S$	2	0	1	2
$\bullet$	2	3	0	1

.

Figure 4: The left and middle functions satisfy (Hex') and (Hex"). The right one satisfies (Hex') but not (Hex").

is commented) as for instance $\mathbf{Z}/4\mathbf{Z}$. Therefore, the left table satisfies (Hex"). On the middle table below we have swapped the two last rows. We can notice that the function is not longer symmetric. However, we stress that (Ind), (Hex") and (Hex') remain. In (Hex'), given a balanced decomposition (μ_0, μ_1) we have for $\mathbf{P}((X_i, Y_i) = (a, b)) = \mathbf{P}((X_i, Y_i) = (b, a))$ for any pair (a, b) and $i = 0, 1$. Therefore, we conserve (Hex') if we swap the values of $f(a, b)$ and $f(b, a)$. This is what we did for $(\S, \star)$ between the middle and the right table. As commented above (Ind) (and (Hex")) are no longer true after this operation.

Example 4.12. We consider $G = \mathbf{Z}/3Z \times \mathbf{Z}/4\mathbf{Z}$ with $f : (x, y) \mapsto x^{-1}y$. For $A = \{(0,0), (0,1), (0,2), (1,1), (1,2), (1,3)\}$ and its complementary set the random variables F_i are distributed as follows:

v	$(0,0)$	(0,2)	$(0,1)$ and $(0,3)$	$(1,0)$ and $(2,0)$
$\mathbf{P}(F_i = v)$	$6/36$	$4/36$	$4/36$	$2/36$
v	$(1,1)$ and $(2,3)$		$(1,2)$ and $(2,2)$	$(1,3)$ and $(2,1)$
$\mathbf{P}(F_i = v)$	$3/36$		$2/36$	$2/36$

For the set of generators $\{(\pm 1, 0), (0, \pm 1)\}$ this corresponds to the following distribution of the distance:

r	0	1	2	3
$\mathbf{P}(D_i = r)$	$6/36$	$12/36$	$10/36$	$8/36$

.

One can check that for $A' = \{(0,0), (0,1), (0,2), (0,3), (1,1), (1,3)\}$ and its complementary set the same distribution of the distance is obtained as for A. However, the distribution of F_i is different and permits to distinguish A from A'. In particular $\mathbf{P}(F_i = (0,2))$ becomes $6/36 \neq 4/36$.

Example 4.13. In Example 4.9 the inverse implication is not satisfied. The following function on $\{\star, \#, \S, \bullet\}$ is antisymmetric (with involution $i(m) = 4 - m$). Let us first check that it satisfies (Hex'). We have $\mathbf{P}((X_i, Y_i) = (a, b)) = \mathbf{P}((X_i, Y_i) = (b, a))$ so that $\mathbf{P}(F_i = m) = \mathbf{P}(F_i = i(m))$ for every m. Thus we can see f as a symmetric function with random value $\tilde{f}(X, Y) = \{f(X, Y), f(X, Y)\}$ composed with the random choice between the two elements (if they are different). However (Ind) fails. For instance $\mathbf{P}(F = 3|\ X = \star) = 1/4 \neq 0 = \mathbf{P}(F = 3|\ X = \#)$. This illustrates that the implication (Ind)$\Rightarrow$(Hex') in Example 4.9 is not an equivalence.

	$\star$	$\#$	$\S$	$\bullet$
$\star$	0	1	2	3
$\#$	-3	0	1	2
$\S$	-2	-3	0	1
$\bullet$	-1	-2	-3	0

Example 4.14. To take Cayley table of a group is only one way to have X, Y and F pairwise independent. In the discrete setting it was already mentioned for instance in [9] that any latin square gives rise to the hexachordal theorem. Here is such a square with six symbols. 'Latin' stands here for the fact that each symbol appears once and only one on each row and column. An rather simple argument in the theory of latin squares/quasigroups tells that if the following table were a Cayley table it would also be the Cayley table of a group where the neutral element would be both left and up in the upper and left header, respectlively. Here it means that $\heartsuit$ could be considered as the neutral and the elements on the first row and column are the ones of the two headers. Notice that they appear in the same order such that the elements on the diagonal should be their square. Since on the diagonal $\heartsuit$ only appears once, the group can neither be $\mathbf{Z}/6\mathbf{Z}$, $(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z})$ nor $\mathfrak{S}(3)$ where there are at least two elements of order 2.

$\heartsuit$	$\square$	$\triangle$	$\clubsuit$	$\diamondsuit$	$\spadesuit$
$\square$	$\triangle$	$\diamondsuit$	$\spadesuit$	$\heartsuit$	$\clubsuit$
$\triangle$	$\clubsuit$	$\square$	$\diamondsuit$	$\spadesuit$	$\heartsuit$
$\clubsuit$	$\spadesuit$	$\heartsuit$	$\square$	$\triangle$	$\diamondsuit$
$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\triangle$	$\clubsuit$	$\square$
$\spadesuit$	$\diamondsuit$	$\clubsuit$	$\heartsuit$	$\square$	$\triangle$

Also for continuous examples one does not need a group structure. It was already observed in [15] that the group of octonions of modulus 1, identified with $\mathbf{S}^7$ host a hexachordal theorem even though the associativity axiom fails on $\mathbf{S}^7$.

Remark 4.15 (Patterson function). Some papers are considering the setting of separable groups with a bi-invariant Haar probability measure μ (the normalized counting measure for a finite group or, more generally, the one presented in Examples 4.7 and 4.8) and introduce the Patterson function of $A \subset \mathfrak{X}$ defined by $\mathrm{Pat}_A : g \in \mathfrak{X} \mapsto \mu(A \cap g \cdot A)$. They also reformulate Babbitt's theorem as $\mathrm{Pat}_A = \mathrm{Pat}_{A^c}$ for every A of measure 1/2. Let us explain that this is an equality of densities (with respect to μ) that corresponds to our equality of measures (Hex). Note first that our formulation with measures is justified by the fact that in general $F = f(X, Y)$ does not possess a density with respect to μ. Let (X, Y, F) be as in Examples 4.8 with $F = X^{-1}Y$. The three components are pairwise independent of law μ. Similar to Remark 2.1 we have the following rewriting of (Hex) for groups

$$\mathbf{P}(F \in E|\ X \in A \text{ and } Y \in A) = \mathbf{P}(F \in E|\ X \in A^c \text{ and } Y \in A^c). \tag{7}$$

The left-hand side also writes $4\mathbf{P}(F \in E \text{ and } X \in A \text{ and } X \cdot F \in A)$. Since F and X are independent this is

$$4 \int_E \mathbf{P}(X \in A \text{ and } X \cdot g \in A)\ d\mu(g) = \int_E 4\mathrm{Pat}_A(g^{-1}) d\mu(g)$$

Using the left invariance of μ in the definition of Pat_A we finally see that the law of F conditional upon ($X \in A$ and $Y \in A$) admits the density $4\mathrm{Pat}_A$ with respect to μ. One can proceed identically for the right-hand side of (7) so that for every E measurable $\mathbf{P}(F \in E|\ X \in A^c \text{ and } Y \in A^c) = \int_E 4\mathrm{Pat}_{A^c}(g)d\mu(g)$. From (7) It follows the equality of the two Patterson functions at almost every $g \in \mathfrak{X}$. As proved in [8, 21, 6, 15] this identity in fact holds not only for almost every but for every $g \in \mathfrak{X}$.

Acknowledgment.

The authors wish to thank the colleagues at IRMA-University of Strasbourg for all the informal discussions that contributed to the clarification of many ideas developed in this article.

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