ABSOLUTE CONTINUITY OF WASSERSTEIN GEODESICS IN THE HEISENBERG GROUP

A.FIGALLI AND N.JUILLET

ABSTRACT. In this paper we answer to a question raised by Ambrosio and Rigot [2] proving that any interior point of a Wasserstein geodesic in the Heisenberg group is absolutely continuous if one of the end-points is. Since our proof relies on the validity of the so-called Measure Contraction Property and on the fact that the optimal transport map exists and the Wasserstein geodesic is unique, the absolute continuity of Wasserstein geodesic also holds for Alexandrov spaces with curvature bounded from below.

1. INTRODUCTION

The optimal transportation problem is nowadays a very active research domain. After having being intensively studied in a Euclidean and a Riemannian setting by many authors, it has been recently investigated also in a sub-Riemannian framework. In particular, optimal transportation in the Heisenberg group $\mathbb{H}^n$ has been first studied by Ambrosio and Rigot [2], where it is proved that the Monge problem can been solved, and a Brenier-McCann representation holds (see Proposition 1.1).

The books by Villani [12, 13] provide an excellent presentation of optimal mass transportation, while two general references about the Heisenberg group are the books by Montgomery [6] and the one by Capogna, Danielli, Pauls and Tyson [4]. The reader is referred to these books for a detailed presentation on these two active mathematical domains.

The aim of this paper is to study the absolute continuity of Wasserstein geodesics, and answer to an open problem proposed by Ambrosio and Rigot [2, Section 7(c)]. Before stating our result in Theorem 1.2, we briefly introduce the concepts appearing in this paper.

Let n be a non-negative integer. The *Heisenberg group* $\mathbb{H}^n$ can be written in the form $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$, and an element of $\mathbb{H}^n$ is written as $(z;t) = (z_1, \cdots, z_n; t)$. The group structure of $\mathbb{H}^n$ is given by

$$(z_1, \cdots, z_n; t) \cdot (z'_1, \cdots, z'_n; t') = \left(z_1 + z'_1, \cdots, z_n + z'_n; t + t' + 2\sum_{k=1}^{n} \Im(z_k \overline{z'_k}) \right),$$

where $\Im(z)$ denotes the imaginary part of a complex number. With this structure, $\mathbb{H}^n$ is a Lie group (with neutral element $0_{\mathbb{H}} = (0_{\mathbb{C}^n}; 0)$). As basis for the associated Lie Algebra of left-invariant vector fields we take as usual $\big(\mathbf{X}_1, \cdots, \mathbf{X}_n, \mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{T}\big)$, where

$$\begin{aligned} \mathbf{X}_k &= \partial_{x_k} + 2y_k\partial_t \qquad \text{for } k = 1, \dots, n \\ \mathbf{Y}_k &= \partial_{y_k} - 2x_k\partial_t \qquad \text{for } k = 1, \dots, n \\ \mathbf{T} &= \partial_t. \end{aligned}$$

The horizontal distribution $(\mathbf{X}_1, \cdots, \mathbf{X}_n, \mathbf{Y}_1, \cdots, \mathbf{Y}_n)$ allows to define a sub-Riemannian distance, called *Carnot-Carathéodory distance*, that we denote by d_C. This distance restricts to Euclidean lines l of $\mathbb{R}^{2n+1}$ as follows. If for each point $p \in l$ the direction of the line l at p is spanned by the horizontal distribution, then the restriction of d_C to l equals up to a constant the Euclidean distance d_{Euc}. If it is not, then there is a constant C and a real function $F(s) = Cs^{1/2} + o(s^{1/2})$ as $s \downarrow 0$ such that $d_C(\cdot, p)|_l = F(d_{Euc}(\cdot, p))$. In particular, the restriction of d_C on lines directed by $\mathbf{T}$ is $\sqrt{\pi} d_{Euc}^{1/2}$. Inspired by the exponential map in Riemannian geometry, Ambrosio and Rigot introduced in [2] a special exponential map $\exp_{\mathbb{H}}$, which differs from the isomorphism between the Lie algebra and the Lie group: the numbers $A + \mathbf{i}B \in \mathbb{C}^n$ and $w \in [-2\pi, 2\pi]$ parameterize the geodesics starting from $0_{\mathbb{H}}$ which can be written as $s \mapsto \exp_{\mathbb{H}}(s(A + \mathbf{i}B), sw)$.

The Monge-Kantorovich problem (with a quadratic cost) is the following: given μ_0 and μ_1 two probability measures on a complete and separable metric space (X, d), minimize

$$\inf_{\pi} \int_{X \times X} d(p,q)^2 \, d\pi(p,q)$$

among all couplings π of μ_0 and μ_1 (that is, among all probability measures π on $X \times X$ whose marginals are μ_0 and μ_1). The square root of the above infimum (which indeed is a minimun) gives rise to a distance on the so-called Wasserstein space $W_2(X) = \{\mu \mid \int_X d^2(x_0, x) \, d\mu(x) < \infty\}$. It turns out that if (X, d) is length space, $W_2(X)$ is also length space. In this paper we will investigate the absolute continuity of measures staying in a geodesic path from an absolutely continuous measure to an other measure of $\mathbb{H}^n$. Proposition 1.1 proved by Ambrosio and Rigot provides a nice representation of such geodesics using the notion of *approximate differential*, see [1, Definition 5.5.1].

We recall that $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ has an approximate differential at $x \in M$ if there exists a function $h : \mathbb{R}^{2n+1} \to \mathbb{R}$ differentiable at x such that the set $\{f = h\}$ has density 1 at x with respect to the Lebesgue measure. In this case the approximate derivatives of f at x are defined as

$$\begin{aligned}(\tilde{\mathbf{X}} f(x) + \mathbf{i}\tilde{\mathbf{Y}} f(x), \tilde{\mathbf{T}} f(x)) &:= (\mathbf{X}h(x) + \mathbf{i}\mathbf{Y}h(x), \mathbf{T}h(x)) \\ &= (\mathbf{X}_1 h(x) + \mathbf{i}\mathbf{Y}_1 h(x), \ldots, \mathbf{X}_n h(x) + \mathbf{i}\mathbf{Y}_n h(x), \mathbf{T}h(x)).\end{aligned}$$

It is not difficult to show that this definition makes sense.

Proposition 1.1. [2, Theorem 5.1 and Remark 5.9] *Let μ_0 and μ_1 be two Borel probability measures on $\mathbb{H}^n$. Assume that μ_0 is absolutely continuous with respect to $\mathcal{L}^{2n+1}$ and that*

$$\int_{\mathbb{H}^n} d_C(0_{\mathbb{H}}, x)^2 d\mu_0(x) + \int_{\mathbb{H}^n} d_C(0_{\mathbb{H}}, y)^2 d\mu_1(y) < +\infty.$$

Then there exists a unique optimal transport plan from μ_0 to μ_1. Moreover, there exists a function φ which is approximately differentiable μ_0-a.e. such that the optimal transport plan is concentrated on the graph of

$$T(x) := x \cdot \exp_{\mathbb{H}}(-\tilde{\mathbf{X}}\varphi(x) - \mathbf{i}\tilde{\mathbf{Y}}\varphi(x), -\tilde{\mathbf{T}}\varphi(x)).$$

As a consequence of this theorem, it is observed in [2, Section 7 (c)] that the family of measures

$$\mu_s := T_{s\#}\mu \quad \text{with} \quad T_s(x) := x \cdot \exp_{\mathbb{H}}(-s\tilde{\mathbf{X}}\varphi(x) - \mathbf{i}s\tilde{\mathbf{Y}}\varphi(x), -s\tilde{\mathbf{T}}\varphi(x))$$

with $s \in [0,1]$ is a constant-speed geodesic in $W_2(\mathbb{H}^n)$ between μ_0 and μ_1. Moreover, since φ is approximately differentiable μ_0-a.e., a simple variant of the proof of [2, Lemma 4.7] shows that $T(x)$ does not belong to the cut-locus of x for μ_0-a.e. x. In particular this implies that the geodesic in $W_2(\mathbb{H}^n)$ between μ_0 and μ_1 is unique.

In [2, Section 7 (c)] the following open problem is raised: are all measures μ_s absolutely continuous for $s \in [0,1)$?

This question is motivated by the fact that the above property holds in the Euclidean and the Riemannian setting (see [13, Chapter 8]). The aim of this paper is to give a positive answer to the above question.

Since the Heisenberg group is non-branching, by [13, Theorem 7.29] we know that for any time $s \in [0,1)$ the map T_s is μ_0-essentially injective (i.e. its restriction to a set with full μ_0-measure is injective), and there exists an inverse transport map S_s uniquely defined up to μ_s-negligible sets such that $S_s \circ T_s = \mathrm{Id}$ μ_0-a.e. (and so $S_{s\#}\mu_s = \mu_0$).

Our main result is the following:

Theorem 1.2. *Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of the Wasserstein space $W_2(\mathbb{H}^n)$ and assume that μ_0 has density ρ with respect to $\mathcal{L}^{2n+1}$. Then for any $s \in [0,1)$ the measure μ_s is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{2n+1}$, and its density is bounded by*

$$\frac{1}{(1-s)^{2n+3}}\rho \circ T_s^{-1}|_{T_s(A)},$$

where T_s is the (μ_0-almost uniquely defined) optimal transport map from μ_0 to μ_s, and A is any set of full μ_0-measure on which T_s is injective.

We remark that the usual way to prove the absolute continuity of the intermediate measures is to use the Monge-Mather shortening principle (see [13, Chapter 8]). In Section 2 we will see that this approach cannot work for the Heisenberg group. We will also give an example of an optimal transport $(\mu_t)_{t\in[0,1]}$ such that the measure at time $1/2$ is concentrated on a set of Hausdorff dimension 1, while the sets of dimension 1 are negligible for μ_0 and μ_1. These "bad" results show that strange phenomena can occour in the Heisenberg case, and this made less clear the answer to absolute continuity question.

However, in Section 3 we will see that the absolutely continuity is a consequence of the following two properties: the so-called *MCP* (Measure Contraction Property), which is indeed true in the Heisenberg group [5], and the fact that the optimal transport map exists and the Wasserstein geodesic is unique.

Thanks to this fact, we observe that the same proof of the absolute continuity can be done in Alexandrov spaces with a lower curvature bound. Indeed, in this case the existence of an optimal transport map and the uniqueness of the Wasserstein geodesic were proved by Bertrand [3] under the assumption that μ_0 is compactly supported and absolutely continuous with respect to the Hausdorff measure. Moreover, the *MCP* property holds, see [7, Lemma 2.3 and Proposition 2.8]. Therefore we obtain the following result (see also Remark 2.1):

Theorem 1.3. *Let (X,d) be an n-dimensional, complete Alexandrov space with curvature $\geq K$. Let μ_0 and μ_1 be two compactly supported probability measures, with μ_0 absolutely continuous with respect to the n-dimensional Hausdorff measure $\mathcal{H}^n_d$. Denote by μ_s the unique Wasserstein geodesic between μ_0 and μ_1. Then, for* *any $s \in [0,1)$, the measure μ_s is absolutely continuous with respect to $\mathcal{H}_d^n$, and its density is bounded by*

$$\frac{1}{1-s}\left(\frac{s_{K(n-1)}\left(\frac{d(x,T_s^{-1}(x))}{s\sqrt{n-1}}\right)}{s_{K(n-1)}\left((1-s)\frac{d(x,T_s^{-1}(x))}{s\sqrt{n-1}}\right)}\right)^{n-1}\rho\circ T_s^{-1}(x)|_{T_s(A)}.$$

Here T_s is the (μ_0-almost uniquely defined) optimal transport map from μ_0 to μ_s, A is any set of full μ_0-measure on which T_s is injective, and the function $s_R(t)$ is given by

$$s_R(t) := \begin{cases} \frac{1}{\sqrt{R}}\sin(\sqrt{R}t) & \text{if } R>0,\\ t & \text{if } R=0,\\ \frac{1}{\sqrt{-R}}\sinh(\sqrt{-R}t) & \text{if } R<0.\end{cases}$$

2. Failure of the Monge-Mather shortening principle

A good presentation of the Monge-Mather shortening principle can be found in [13, Chapter 8]. We give here a simplified picture of it in the particular case of geodesic spaces.

Let (X,d) be a geodesic space, and denote by $\mathcal{H}_d$ the Hausdorff measure (here, we do not care about the dimension of the Hausdorff measure). The idea of the shortening lemma is the following: fix a Borel set K, and take 4 points $a,b,p,q \in K$. Suppose that we want to transport a and b on p and q (this is an informal way to say that we want to transport the measure $\frac{1}{2}(\delta_a+\delta_b)$ onto $\frac{1}{2}(\delta_p+\delta_q)$), and assume that for the quadratic cost it is optimal to send a on p and b on q, that is

$$d^2(a,p)+d^2(b,q) \le d^2(a,q)+d^2(b,p).$$

Consider now two constant-speed geodesics $\alpha,\beta:[0,1]\to X$ from a to p and from b to q respectively, and suppose that we can prove the following estimate: there is a constant $C(K,s)$ (depending only on K and on the time $s\in[0,1]$) such that

$$C(K,s)d(\alpha(s),\beta(s)) \ge d(a,b).$$

Then, given any Wasserstein geodesic $(\mu_s)_{s\in[0,1]}$ contained in K, if μ_0 is absolutely continuous with respect to $\mathcal{H}_d$ one can easily prove that also μ_s is absolutely continuous with respect to $\mathcal{H}_d$.

The Heisenberg group $(\mathbb{H}^n, d_C)$ with the Lebesgue measure can be put in the above framework.

2.1. Horizontal right translations as optimal transport.

The Lebesgue measure $\mathcal{L}^{2n+1}$ is the Haar measure of the Heisenberg group because the left translations of $\mathbb{H}^n$ are affine transformation with determinant 1. The $(2n+2)$-dimensional Hausdorff measure is also a Haar measure because d_C is left-invariant and $2n+2$ is the correct dimension. Then by unicity, both measures are equal up to a constant.

We recall that right translations by an horizontal vector provide an optimal transport in the Heisenberg group. This can be proved projecting everything on $\mathbb{C}^n$ and comparing any transport with the optimal Euclidean transport (which indeed is a translation), see also [2, Example 5.7].

Let μ_0 be the restriction of $\mathcal{L}^{2n+1}$ to $(0,1)^{2n+1}$, and consider the horizontal vector $u=(1,0,\dots,0;0)$. With the notation of the introduction, T_s is given for any $s \in [0,1]$ by the map $a \mapsto a \cdot (s,0,\ldots,0;0)$. More precisely, writing a as $(x+\mathbf{i}y, z_2, \ldots, z_n; t)$, we have

$$T_s(a) = ((x+s)+\mathbf{i}y, z_2, \ldots, z_n; t+2sy). \tag{1}$$

We observe that T_s is affine on $\mathbb{R}^{2n+1}$ with Jacobian determinant 1, so that the measure $\mu_s = T_{s\#}\mu_0$ is absolutely continuous. However, as we will show, the shortening principle does not hold.

Fix $a \in (0,1)^{2n+1}$, and let

$$a_\varepsilon := a + \varepsilon(\mathbf{i}, 0, \ldots, 0; -2x-4s) = (x+\mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; t - 2\varepsilon x - 4\varepsilon s)$$

with ε small enough so that $a_\varepsilon \in (0,1)^{2n+1}$. Then, using (1) twice,

$$\begin{aligned} T_s(a_\varepsilon) &= a_\varepsilon \cdot (s, \ldots, 0; 0) \\ &= ((x+s)+\mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; (t-2\varepsilon x - 4\varepsilon s) + 2s(y+\varepsilon)) \\ &= ((x+s)+\mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; (t+2sy) - 2\varepsilon(x+s)) \\ &= T_s(a) \cdot v_\varepsilon \end{aligned}$$

where v_ε is the horizontal vector $(\mathbf{i}\varepsilon, 0, \ldots, 0; 0)$. Therefore

$$d_C(a, a_\varepsilon) = d_C(0_{\mathbb{H}}, p^{-1} \cdot a_\varepsilon) = d_C\left(0_{\mathbb{H}}, (\mathbf{i}\varepsilon, 0, \ldots, 0; -4\varepsilon s)\right) \sim 2\sqrt{\pi|\varepsilon| s}$$

as $\varepsilon \to 0$, while

$$d_C(T_s(a), T_s(a_\varepsilon)) = d_C(0, v_\varepsilon) = |\varepsilon|.$$

Thus we see that the shortening principle cannot hold. Moreover from this example one can also see that there is no hope to find a decomposition of $(0,1)^{2n+1}$ into a family of countable Borel sets such that on each set the shortening principle holds, possibly with a different constant (if such weaker condition holds, one can still prove quite easily the absolute continuity of the interpolation).

2.2. **An instructive optimal transport.** We consider the following transportation problem: the two measures μ_0 and μ_1 are concentrated on the vertical line

$$L := \{(z;t) \in \mathbb{H}^n \mid z = 0_{\mathbb{C}^n})\},$$

with μ_0 concentrated on the negative part $L^- = L \cap \{t \le 0\}$ and μ_1 on the positive one $L^+ = L \cap \{t \ge 0\}$. We remark that the restriction of the quadratic cost d_C^2 on L is linear in the real coordinate, that is

$$d_C^2\left((0_{\mathbb{C}^n}; t), (0_{\mathbb{C}^n}; t')\right) = \pi|t-t'|.$$

We can then reduce the transportation problem to a L^1-Kantorovich-Rubinstein problem on the real line $\mathbb{R}$. This situation is quite particular because all couplings of μ_0 and μ_1 are optimal (see [12, Chapter 2]).

Let us investigate a concrete example: identifying $L = \{0_{\mathbb{C}^n}\} \times \mathbb{R}$ with $\mathbb{R}$, let μ_0 and μ_1 be $\mathcal{L}^1\lfloor[-1,0]$ and $\mathcal{L}^1\lfloor[0,1]$ respectively. A (optimal) coupling is given by $(\mathrm{Id}, T)_\#\mu_0$, where the transport map is $T : (0_{\mathbb{C}^n}; t) \mapsto (0_{\mathbb{C}^n}; -t)$.

There is a multiple choice of geodesics between $(0_{\mathbb{C}^n}; t)$ and $T(0_{\mathbb{C}^n}; t)$ (actually the cut-locus of a point $p \in L$ is exactly $L \setminus \{p\}$). To construct a Wasserstein geodesic, we select the (unique) geodesic between $(0_{\mathbb{C}^n}; t)$ and $(0_{\mathbb{C}^n}; -t)$ whose midpoint is on the horizontal half-line $\{(r, 0, \ldots, 0; 0) \mid r \in [0, +\infty)\}$. This midpoint is exactly $(\sqrt{\frac{2|t|}{\pi}}, 0, \ldots, 0; 0)$.

Using these geodesics, we define a Wasserstein geodesic $(\mu_s)_{s\in[0,1]}$ between μ_0 and μ_1 which satisfies the following property: although μ_0 and μ_1 are absolutely continuous with respect to the 2-dimensional Hausdorff measure (induced by the distance d_C), the intermediate measure $\mu_{1/2}$ is concentrated on the horizontal line $\{(r, 0, \ldots, 0; 0) \mid r \in \mathbb{R}\}$ whose dimension is 1. This observation could suggest that one can find a measure μ_0 absolutely continuous with respect to the Lebesgue measure such that $\mu_{1/2}$ is not absolutely continuous because concentrated on a set of lower dimension. As announced in the introduction, we will prove in Section 3 that this cannot happen.

Remark 2.1. As explained in the book by Villani [13, Notes on Chapter 8], it can be proved that the shortening lemma holds for non-negatively curved Alexandrov spaces (this follows from an estimate found by the first author, see [13, Equation (8.45)]). It is not known if the property is also true for Alexandrov spaces with curvature bounded from below, see [13, Open Problem 8.21].

3. Proof of Theorem 1.2

The starting point for the proof of the theorem is an estimate of the second author on the size of a set when contracted along geodesics to a point [5]. Given $x, y \in \mathbb{H}^n$ and $s \in (0, 1)$, let us denote by $\mathcal{M}_s(x, y)$ the set of points m such that

$$d_C(x, m) = s d_C(x, y), \qquad d_C(m, y) = (1 - s) d_C(x, m).$$

For $E \subset \mathbb{H}^n$, we denote by $\mathcal{M}_s(E, y)$ the set

$$\mathcal{M}_s(E, y) := \bigcup_{x \in E} \mathcal{M}_s(x, y).$$

We remark that, for fixed y, for $\mathcal{L}^{2n+1}$-a.e. x the set $\mathcal{M}_s(x, y)$ is a single point and the curve $s \mapsto \mathcal{M}_s(x, y)$ is the unique constant-speed geodesic between x and y.

Proposition 3.1. [5, Section 2] *Let $y \in \mathbb{H}^n$ and E a measurable set. Then $\mathcal{M}_s(E, y)$ is measurable and for any $s \in [0, 1]$,*

$$\mathcal{L}^{2n+1}\big(\mathcal{M}_s(E, y)\big) \geq (1 - s)^{2n+3} \mathcal{L}^{2n+1}(E).$$

Remark 3.2. This estimate, in a more elaborate form, is known as $MCP(0, 2n + 3)$. On Riemannian manifolds this property is shown to be equivalent to a Ricci curvature bound, and it can be regarded as a generalized notion of a lower Ricci curvature bound for metric measure spaces [7, 11]. This notion is however different from the Curvature-Dimension condition $CD(K, N)$ introduced by Lott-Villani [8, 9] and Sturm [10, 11], and is weaker if the metric space is non-branching. In particular $CD(K, N)$ does not hold in $\mathbb{H}^n$ for any curvature K and any dimension N (see [5]).

The idea of the proof is now the following: first we approximate the target measure μ_1 by a sequence of discrete measures, and using Proposition 3.1 we prove the absolute continuity of the interpolation in the case of a discrete target measure. Then we pass to the limit, and we finally get the upper bound on the density of the interpolation.

Let $\mu_1^k = \frac{1}{k} \sum_{i=1}^k \delta_{y_i}$ be a sequence weakly converging to μ_1, and denote by T^k the optimal transport map between $\mu_0 = \rho \mathcal{L}^{2n+1}$ and μ_1^k. As in the introduction, $(\mu_s^k)_{s \in [0,1]}$ denotes the unique Wasserstein geodesic between μ_0 and μ_1^k, and T_s^k is the transport map from μ_0 to μ_s^k.

We remark that, if we prove the estimate with a certain set A of full μ_0-measure, then the bound will obviously be true also for any set containing A. Thus, up to a replacement of A with $A \cap \{\rho > 0\}$, we can assume that $A \subset \{\rho > 0\}$, so that μ_0 and $\mathcal{L}^{2n+1}$ are equivalent on A.

For each $i = 1, \ldots, k$, let $A_i^k \subset A$ be the set of points $x \in A$ such that $T^k(x) = y_i$. The sets A_i^k are mutually disjoint and $\mu_0\big(\mathbb{H}^n \setminus \cup_{i=1}^k A_i^k\big) = 0$. Denoting by m_i^k the restriction of μ_0 to A_i, we observe that the measures m_i^k are mutually singular, $\mu_0 = \sum_{i=1}^k m_i^k$, and $T^k_{\#} m_i^k = \frac{1}{k}\delta_{y_i}$.

Let us fix i. Since $T^k(A_i) = y_i$, the curve $s \mapsto T_s^k(x)$ is the unique geodesic from x to y_i for $\mathcal{L}^{2n+1}$-a.e. $x \in A_i$. Therefore there exists $B_i \subset A_i$ such that $\mathcal{L}^{2n+1}(A_i \setminus B_i) = 0$ and $s \mapsto T_s^k(x)$ is the unique geodesic from x to y_i for all $x \in B_i$. Consider now $E \subset B_i$. By the uniqueness of the geodesics from E to y_i we have

$$\mathcal{M}_s(E, y_i) = T_s^k(E).$$

We can therefore apply Proposition 3.1 to obtain that, for any $E \subset B_i$

$$\mathcal{L}^{2n+1}\big(T_s^k(E)\big) \geq (1-s)^{2n+3}\mathcal{L}^{2n+1}(E).$$

Since $\mathcal{L}^{2n+1}(A_i \setminus B_i) = 0$, the above estimate is still true if $E \subset A_i$. Recalling now that the sets A_i are disjoint and T_s^k is essentially injective, we easily obtain

$$\forall E \subset A, \qquad \mathcal{L}^{2n+1}\big(T_s^k(E)\big) \geq (1-s)^{2n+3}\mathcal{L}^{2n+1}(E)$$

Indeed it suffices to take $E \subset A$, split it as $E_i = E \cap A_i$, write the estimate for E_i and add all the estimates for $i = 1, \ldots, k$. The above property can also be stated by saying that, for any $F \subset T_s^k(A)$,

$$\mathcal{L}^{2n+1}\big(F\big) \geq (1-s)^{2n+3}\mathcal{L}^{2n+1}((T_s^k)^{-1}(F) \cap A),$$

or equivalently

$$\int_A g(T_s^k(x))\, d\mathcal{L}^{2n+1}(x) \leq \frac{1}{(1-s)^{2n+3}} \int_{\mathbb{H}^n} g(y)\, d\mathcal{L}^{2n+1}(y) \tag{2}$$

for all $g \in C_c(\mathbb{H}^n)$, with $g \geq 0$. Since the Wasserstein geodesic between μ_0 and μ_1 is unique, by the stability of the optimal transport we have that, for any fixed s, the sequence μ_s^k weakly converges to μ_s, and the optimal transport maps T_s^k from μ_0 to μ_s^k converge in μ_0-measure to T_s from μ_0 to μ_s (see [13, Chapter 7 and Corollary 5.21]).

Thus, up to a subsequence, we can assume that $T_s^k \to T_s$ μ_0-a.e., which in particular implies that $T_s^k \to T_s$ for $\mathcal{L}^{2n+1}$-a.e. $x \in A$. We can therefore pass to the limit in (2), obtaining

$$\int_A g(T_s(x))\, d\mathcal{L}^{2n+1}(x) \leq \frac{1}{(1-s)^{2n+3}} \int_{\mathbb{H}^n} g(y)\, d\mathcal{L}^{2n+1}(y) \tag{3}$$

for all $g \in C_c(\mathbb{H}^n)$, $g \geq 0$. Moreover, arguing by approximation and using the Monotone Convergence Theorem, we obtain that (3) holds for any measurable function $g \geq 0$ (in this case, both sides of the equation can be infinite).

From this fact we can directly conclude that T_s sends a set with positive Lebesgue measure into a set with positive Lebesgue measure, which implies that μ_s is absolutely continuous.

In order to prove the bound on the density of μ_s, we consider in (3)

$$g(y) := \chi_{T_s(A)}(y) h(y) \rho \circ T_s^{-1}(y),$$

with $h \geq 0$. In this way we get

$$\begin{aligned}\int_{T_s(A)} h(y)\,d\mu_s(y) &= \int_A h(T_s(x))\,d\mu_0(x)\\ &= \int_A h(T_s(x))\rho(x)\,d\mathcal{L}^{2n+1}(x)\\ &\leq \frac{1}{(1-s)^{2n+3}}\int_{\mathbb{H}^n} h(y)\rho\circ T_s^{-1}(y))\,d\mathcal{L}^{2n+1}(y).\end{aligned}$$

From the arbitrariness of h and the fact that μ_s is concentrated on $T_s(A)$ the bound follows.

References

[1] L.Ambrosio, N.Gigli & G.Savaré: *Gradient flows in metric spaces and in the Wasserstein space of probability measures.* Lectures in Mathematics, ETH Zurich, Birkhäuser Verlag, 2005.

[2] L.Ambrosio & S.Rigot: *Optimal mass transportation in the Heisenberg group.* J. Funct. Anal., **208** (2004), no. 2, 261–301.

[3] J.Bertrand: *Optimal transport on Alexandrov spaces.* Preprint.

[4] L.Capogna, D.Danielli, S.D.Pauls & J.T.Tyson: *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Progress in Mathematics, Basel, Birkhäuser Verlag, 2007.

[5] N.Juillet: *Ricci curvature bounds and geometric inequalities in the Heisenberg group.* Preprint SFB611 Bonn.

[6] R.Montgomery: *A tour of subriemannian geometries, their geodesics and applications.* Mathematical Surveys and Monographs, **91**, American Mathematical Society, Providence, RI, 2002.

[7] S.Otha: *On the measure contraction property of metric measure spaces.* Preprint.

[8] J.Lott & C.Villani: *Ricci curvature for metric-measure spaces via optimal transport.* Annals of Math., to appear.

[9] J.Lott & C.Villani: *Weak curvature conditions and functional inequalities.* J. Funct. Anal., **245** (2007), no. 1, 311–333.

[10] K.-T.Sturm: *On the geometry of metric measure spaces. I.* Acta Math., **196** (2006), no. 1, 65–131.

[11] K.-T.Sturm: *On the geometry of metric measure spaces. II.* Acta Math., **196** (2006), no. 1, 133–177.

[12] C. Villani: *Topics in optimal transportation.* Graduate Studies in Mathematics, **58**, American Mathematical Society, Providence, RI, 2003.

[13] C.Villani: *Optimal transport, old and new.* Notes from the Saint-Flour 2005 Summer School. Available online at www.umpa.ens-lyon.fr/~cvillani.