ON DISPLACEMENT INTERPOLATION OF MEASURES INVOLVED IN BRENIER'S THEOREM.

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ABSTRACT. We prove that in the Wasserstein space built over $\mathbb{R}^d$ the subset of measures that does not charge the non-differentiability set of convex functions is not displacement convex. This completes the study of Gigli on the geometric structure of measures meeting the sharp hypothesis of the refined version of Brenier's Theorem.

Optimal transport is nowadays a central tool in many fields of analysis, differential geometry and probability theory (see e.g. the books by Villani [11, 12]). It takes its origin in the problem of Monge, asking for the shortest way to displace an amount of soil from one place of the Euclidean space to a heap of soil at another place. This problem induces a very natural distance between probability measures – the Wasserstein distance – where the measures represent the heaps of soil. Wasserstein distance is somewhat complementary with the Lebesgue L^p norms because it corresponds to an "horizontal" displacement: in first approximation, for two localized probability measures on $\mathbb{R}^d$, the Wasserstein distance is the distance between their barycenters.

Brenier's Theorem [4] on monotone rearrangement of maps of $\mathbb{R}^d$ has become the very core of the theory of optimal transport. It gives a representation of the optimal transport map in term of gradient of convex functions. A very enlightening heuristic on $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is proposed in [7] where it appears with an infinite differential structure and the Wasserstein distance is seen as a Riemannian-like distance. This point of view has raised a lot of developments, among which the gradient flow theory presented in [2]. In [6], Gigli explores the Riemannian-like structure and proposes to think of the measures meeting the hypothesis of a refined version of Brenier's Theorem as the regular points of the Wasserstein space. In this paper we show that the set of those transport-regular measures is not geodesically convex (Theorems 2.2 and 2.5). It is quite surprising because it is well-known that the subset made of absolutely continuous measures (the most notorious transport-regular measures) is geodesically convex. It answers a question suggested by Gigli [6, Remark 2.12].

The paper is built as follows: we first recall some main results on the quadratic Monge(-Kantorovich) problem. The key idea of our result can be found in Lemma 1.6 while Proposition 1.8 gives a way, together with Proposition 1.7, to characterize transport-regular measures. In the second (and last) part, we prove the main theorem (Theorem 2.2) and its generalization (Theorem 2.5) thanks to explicit constructions.

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Remark 0.1. While the paper was already submitted, Shin-Ichi Ohta explained me that Theorem 2.2 together with [6, Corollary 6.6] show that on $(\mathcal{P}_2(\mathbb{R}^d), W_2)$, which is a positively curved space (see [12, Bibliographic notes of Chapter 26]), there is a geodesic curve γ such that $\gamma(0), \gamma(1)$ are regular and the mid-point $\gamma(1/2)$ is non-regular in the sense of Alexandrov spaces. This phenomenon can only occur in infinite dimensional spaces. Indeed it was proven in [10, Corollary of Theorem 1.2(A), p. 132] that in *finite dimensional Alexandrov spaces with curvature bounded below* the tangent cone at $\gamma(1/2)$ is a Hilbert space.

1. Quadratic Monge-Kantorovich problem in $\mathbb{R}^d$.

1.1. **The problem.** We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the Wasserstein space on $\mathbb{R}^d$, i.e., the space of Borel probability measures μ on $\mathbb{R}^d$ such that $\int |x|^2 d\mu(x) < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$. We consider

$$(1.1) \qquad C(\pi) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |p - q|^2 d\pi(p, q)$$

where $\pi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ has first marginal $e_{0\#}\pi = \mu_0$ and second marginal $e_{1\#}\pi = \mu_1$. Here $e_0 \otimes e_1 = \mathrm{Id}_{\mathbb{R}^d \times \mathbb{R}^d}$. We say that π is a transport plan (or a coupling) between μ_0 and μ_1 if it satisfies the conditions on marginals. The Monge-Kantorovich problem that we consider here consists in minimizing $C(\pi)$ among all transport plans (see [12] or [2] for generalizations of this problem). The minimizers in (1.1) are called optimal transport plans. In this setting, it is well-known that there exists at least one optimal transport plan (see e.g. [11, Proposition 2.1.]). The Wasserstein distance between μ_0 and μ_1 is then $W_2(\mu_0, \mu_1) = \sqrt{C(\pi)}$ where π is a minimizer. Among the questions of interest are the uniqueness and the properties of the optimal transport plans. These questions are correlated. Actually if we can prove some special property for minimizers of (1.1), we can restrict the set of candidates and may obtain uniqueness. A key feature of an optimal transport plan is the cyclical monotonicity of its support (which actually is a property that *characterizes* optimal plans). It will be described in subsection 1.2. Continuing the analysis of the support of optimal π's can lead to deterministic couplings, the so-called transport maps presented in subsection 1.3.

1.2. **Cyclical monotonicity.** We first review some basic results. The problem (1.1) can somewhat be discretized by introducing the cyclically monotone sets.

Definition 1.1. A set $S \subset \mathbb{R}^d \times \mathbb{R}^d$ is said cyclically monotone if and only if for any $n \in \mathbb{N}^*$ and any permutation $\sigma \in \Sigma_n$,

$$(1.2) \qquad \sum_{i=1}^{n} |x_k - y_k|^2 \leq \sum_{k=1}^{n} |x_k - y_{\sigma(k)}|^2,$$

where $(x_k, y_k) \in S$ for $k = 1, \ldots, n$ is arbitrary.

The next result provides a geometric characterization of optimal transport plans.

Proposition 1.2. *Let* $\pi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$. *Then* π *is an optimal transport plan between* $\mu_0 = e_{0\#}\pi$ *and* $\mu_1 = e_{1\#}\pi$ *if and only if* $\mathrm{Spt}(\pi)$ *is cyclically monotone.*

A proof can be founded in [2, Theorem 6.1.4]. See also the comments on Chapter 5 in [12] for different general assumptions.

Note that it is not easy to check whether a particular transport plan is cyclically monotone. One can sometimes identify that it is not, thanks to the next classical fact.

Corollary 1.3 (Cycles of length 2). *Let $\pi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ be an optimal transport plan. Then for any (x, y) and (x', y') in $\mathrm{Spt}(\pi)$ we have*

$$(1.3) \qquad \langle \overrightarrow{xx'}, \overrightarrow{yy'} \rangle \geq 0.$$

Proof. Let (x, y) and (x', y') be in $\mathrm{Spt}(\pi)$. Proposition 1.2 indicates that $|x - y|^2 + |x' - y'|^2 \leq |x - y'|^2 + |x' - y|^2$. Writing $y' - x = (y' - y) + (y - x)$ as well as $y' - x' = (y' - y) + (y - x)$ we obtain

$$(1.4) \qquad \begin{cases} 2\langle y' - y, y - x \rangle = |y' - x|^2 - |y' - y|^2 - |y - x|^2 \\ 2\langle y' - y', y - x' \rangle = |y' - x'|^2 - |y' - y|^2 - |y - x'|^2 \end{cases}$$

The result comes from the difference of these relations. $\qquad \square$

We will say that a measure $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is **order-preserving** if (1.3) is satisfied for any (x, y) and (x', y') in $\mathrm{Spt}(\pi)$.

1.2.1. *Application of Corollary 1.3 to $d = 1$.* The next lemma is a variation on the well-known theory of optimal transport in dimension 1. It will be useful in the proof of Lemma 1.6 which is the main innovation of this paper. Note that a measure on $\mathbb{R} \times \mathbb{R}$ is order preserving if and only if $x < x' \Rightarrow y \leq y'$ for every $(x, y), (x', y') \in \mathrm{Spt}(\pi)$.

Lemma 1.4. *For two measures μ_0 and μ_1 in $\mathcal{P}_2(\mathbb{R})$, there is a unique order-preserving transport plan. It is uniquely determined by*

$$\pi(] - \infty, x] \times] - \infty, y]) = \min(F_0(x), F_1(y)),$$

where $F_i(t) = \mu_i(] - \infty, t])$ is the cumulative distribution of μ_i for $i \in \{1, 2\}$.

Remark 1.5. Thanks to Corollary 1.3, the optimal transport plans must be π.

Proof. It is an easy exercise to see that π is the law of the random vector (X, Y) with $X = F_0^{-1}(U)$, $Y = F_1^{-1}(U)$ where U is uniform on $[0, 1]$ and F_i^{-1} is the generalized inverse of F_i for $i \in \{0, 1\}$. Let us recall that

$$F_i^{-1}(u) = \inf\{t \in \mathbb{R} \mid F(t) \geq u\}.$$

Hence, as F_0 and F_1 are non-decreasing, π is order-preserving.

Let now π' be an a priori different optimal transport plan for the same Monge-Kantorovich problem and $H(s, t) = \pi'(] - \infty, s] \times] - \infty, t])$. We have to prove that $H(s, t) = \min(F_0(s), F_1(t))$. The inequality $\leq$ is obvious because $F_0(s) = \pi'(]-\infty, s] \times \mathbb{R})$ and $F_1(t) = \pi'(\mathbb{R} \times]-\infty, t])$. For the other estimation it is enough to prove $(F_0(s) > H(s, t)) \Rightarrow (F_1(t) \leq H(s, t))$. Indeed one can treat $F_1(t) > H(s, t)$ in the same way. So we assume $F_0(s) > H(s, t)$, i.e., $\pi'(] - \infty, s] \times]t, +\infty[) > 0$. Let $(x, y) \in \mathrm{Spt}(\pi')$ so that $x \leq s$ and $y > t$. Then because of Corollary 1.3, $\mathrm{Spt}(\pi')$ does not meet $]x, +\infty[\times] - \infty, y[\supset]s, +\infty[\times] - \infty, t] =: R$. Therefore $F_1(t) - H(s, t) = \pi'(R) = 0$. $\qquad \square$

1.2.2. *Application of Corollary 1.3 to measures concentrated on curves.* Let $\vec{u}$ be a unit vector of $\mathbb{R}^d$. We define a partial order on $\mathbb{R}^d$ by $p \preceq q$ if and only if

$$\langle \vec{u}, \overrightarrow{pq} \rangle > |q - p| \cos \frac{\pi}{4} \quad \text{or} \quad p = q.$$

Actually $\{q \in \mathbb{R}^d \mid p \preceq q\}$ is an half-cone directed by $\vec{u}$, with edge p and opening angle $\frac{\pi}{4}$. It is easy to check that $\preceq$ is transitive and antisymmetric. We note $p \prec q$ if $p \preceq q$ and $p \neq q$. As in paragraph 1.2.1 we can reinterpret (1.3): let μ_0, μ_1 be two measures of $\mathbb{R}^d$ such that $\preceq$ is a total order on $\mathrm{Spt}(\mu_i)$ for $i = 0, 1$ and $(x, y), (x', y') \in \mathrm{Spt}(\mu_0) \times \mathrm{Spt}(\mu_1)$. If $x \prec x'$ we have

(1.5)
$$\langle \overrightarrow{xx'}, \overrightarrow{yy'} \rangle \begin{cases} > 0 & \text{if } y \prec y' \\ < 0 & \text{if } y' \prec y \\ = 0 & \text{if } y = y'. \end{cases}$$

So a transport plan π between μ_0 and μ_1 is order-preserving if and only if $x \prec x' \Rightarrow y \preceq y'$ on $\mathrm{Spt}(\pi)$. Note that (1.5) is easy to check. For instance, in the case $y \prec y'$ the angle between the vectors is smaller than $\frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{4}$ (see Figure 1).
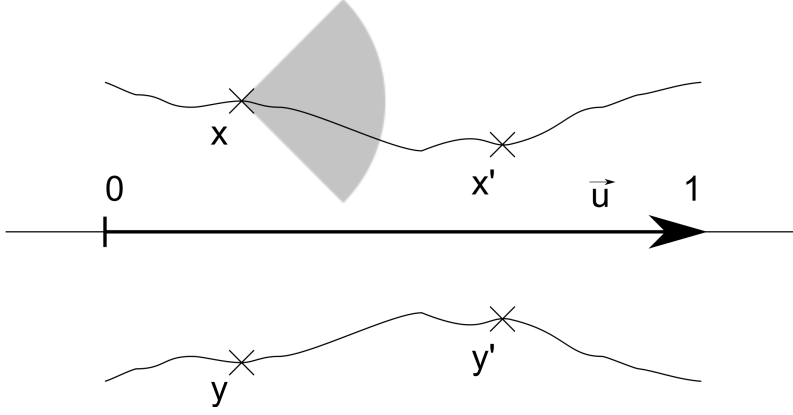

FIGURE 1. Totally ordered sets.

Lemma 1.6. *Let $\vec{u}$ be a unit vector of $\mathbb{R}^d$. Assume that $\mathrm{Spt}(\mu_0)$ and $\mathrm{Spt}(\mu_1)$ are totally ordered with respect to $\vec{u}$. There is a unique order-preserving transport plan π between μ_0 and μ_1. Therefore, thanks to Corollary 1.3, any optimal transport plan between μ_0 and μ_1 is π.*

Proof. Let π be an order-preserving transport plan between the $\mu_i's$. Let p be the orthogonal projection on $L_{\vec{u}}$, the vectorial line directed by $\vec{u}$. The projection p from $(\mathbb{R}^d, \preceq)$ to $(L_{\vec{u}}, \preceq)$ is strictly increasing so that because of Lemma 1.4, $\omega = (p \otimes p)_{\#} \pi$ is an order-preserving transport plan between $p_{\#} \mu_0$ and $p_{\#} \mu_1$ and it is the unique one. In fact, $(L_{\vec{u}}, \preceq, d_{\mathbb{R}^d})$ and $(\mathbb{R}, \leq, d_{\mathbb{R}})$ are isomorphic. But for $i \in \{0, 1\}$ the projection p is one-to-one from $\mathrm{Spt}(\mu_i)$ to its image so that we can deduce π from ω. Explicitly $\pi = (q_0 \otimes q_1)_{\#} \omega$ where q_i is the inverse map of $p : \mathrm{Spt}(\mu_i) \to L_{\vec{u}}$. $\square$

Under the hypothesis of 1.6, with the notation above we have

$$\pi\{(x, y) \mid \langle \vec{u}, \vec{x} \rangle \leq s; \langle \vec{u}, \vec{y} \rangle \leq t\} = \min\left(\mu_0\{\langle \vec{u}, \vec{x} \rangle \leq s\}, \mu_1\{\langle \vec{u}, \vec{y} \rangle \leq t\}\right)$$

and this expression determines uniquely π. This can be seen easily from Lemmas 1.4, 1.6 and their proofs.

1.3. **Necessary and sufficient condition for a Brenier map.** The transport maps from μ_0 to μ_1 are maps $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T_\# \mu_0 = \mu_1$. They are associated to a transport plan by $\pi_T = (\text{Id} \otimes T)_\# \mu_0$. If π_T is a minimizer of (1.1), T is called optimal transport map, or Brenier map. The well known theorem of Brenier ensures the existence of optimal maps under the assumption that μ_0 is absolutely continuous w.r.t. the Lebesgue measure. In [5] this assumption has been weakened to μ_0 *gives* 0 *mass to Lipschitz hypersurfaces* (see also the discussion below). The recent work by Gigli explores the sharpness of hypothesis on μ_0 permitting to have an optimal transport map for any μ_1. Analytic and a geometric characterizations for these measures are given in [6]. We will see in Theorem 2.2 and 2.5 that contrary to absolutely continuous measures, this set of measures is not stable under displacement interpolation.

The next result, especially implication (2)$\Rightarrow$(1) is part of [6]. In what follows, all convex functions are real-valued.

Proposition 1.7 (Refined version of Brenier's Theorem). *Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then the following statements on μ_0 are equivalent:*

(1) *Any convex function ϕ defined on a convex open $U \subset \mathbb{R}^d$ is μ_0-a.e. differentiable.*

(2) *For any measure $\mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a unique solution π to the Monge-Kantorovich problem and it is induced by a transport map T.*

We will call μ_0 satisfying one of these statement a **transport-regular measure** (or simply regular measure as in [6]). The map T can be written as $\nabla \phi$ for some convex function ϕ.

The statement (1) can be replaced by (1'): Any convex function ϕ defined on $\mathbb{R}^d$ is μ_0-a.e. differentiable. Assuming (1'), let ϕ be a convex function defined on U. We can cover U by countably many open balls B_k where the slope of ϕ is bounded. Then

$$\overline{\phi_k} = \sup\{l(x) | \, l \text{ is linear and } l \leq \phi \text{ on } B_k\}$$

is a convex function of $\mathbb{R}^d$ that coincide with ϕ on B_k. Each points of non-differentiability of ϕ is a point of non-differentiability for some $\overline{\phi_k}$. It follows that ϕ is μ_0-a.e. differentiable. Note that in (1), one can also add the points of ∂U as points of non-differentiability. For that, consider that ∂U is the set of non-differentiability of the convex function $x \to d(x, U)$.

Gigli states (1) in a different way in term of c – c hypersurfaces. These objects appear in the work by Zajíček [13] in relation to the non-differentiability set of a convex function and the first time in relation to optimal transport in a footnote of [5]. Actually Zajíček proved in [13] that the set of non-differentiability of a convex function is included in countably many c – c hypersurface and that conversely, such an union is exactly the set of non-differentiability of some function. We will not define c – c hypersurfaces because we use a different result on the set of non-differentiability points. Here is a weak formulation of a result due to Alberti [1].

Proposition 1.8 (Alberti). *Let ϕ be a convex function of $\mathbb{R}^d$. There are $(d-1)$-dimensional submanifolds $M_k \subset \mathbb{R}^d$ of class $\mathcal{C}^2$, and a set N with $\mathcal{H}^{d-1}(N) = 0$ such that ϕ is differentiable outside from $\cup_{k=1}^{+\infty} M_k \cup N$.*

Example 1.9. We give some examples of transport-regular measures. They share the property to be defined by duality as the measures vanishing on special classes of sets F.

- If μ_0 is absolutely continuous (i.e., $\mathcal{H}^d(F) = 0 \Rightarrow \mu_0(F) = 0$) then it is transport-regular. This assumption is the classical assumption of Brenier's Theorem on existence and uniqueness of minimizers in (1.1).
- If $(\dim \mathcal{H}(F) \leq d - 1 \Rightarrow \mu_0(F) = 0)$, then μ_0 is transport-regular. This is the setting of the more usual refined Brenier Theorem, presented in [11, Theorem 2.32]
- A Lipschitz piece is any set $f(E)$ for some $E \subset \mathbb{R}^{d-1}$ and f a Lipschitz continuous map. If
    - μ_0 is absolutely continuous with respect to $\mathcal{H}^{d-1}$;
    - $\mu_0(F) = 0$ for every Lipschitz piece F (i.e., μ_0 is purely non-rectifiable),
    then μ_0 is transport-regular. Any $\mathcal{C}^1$-hypersurface of $\mathbb{R}^d$ is contained in a countable union of Lipschitz pieces.

The measures in the proof of the main theorem are of the type described in the example below.

Example 1.10. If — (i) μ_0 is absolutely continuous with respect to $\mathcal{H}^{d-1}$; (ii) it does not charge hypersurfaces of class $\mathcal{C}^2$ (i.e., μ_0 is $\mathcal{C}^2$-purely non-rectifiable), then μ_0 is transport-regular.

2. Displacement interpolation of transport-regular measures

It is part of the theory to consider the displacement interpolation introduced by McCann in [9]. Let μ_0, μ_1 be such that there exists a unique optimal plan π between them. For any $t \in [0,1]$ we consider $\mu_t = e_{t\#}\pi$ where $e_t(x,y) = tx + (1-t)y$. For the unicity in the definition of μ_t, we assume that the optimal transport plan π is unique. If $\pi = (\mathrm{Id} \otimes T)_{\#}\mu_0$ we have $\mu_t = T_{t\#}\mu_0$ where $T_t = tT + (1-t)\,\mathrm{Id}$. The curve $(\mu_t)_{t \in [0,1]}$ defines the unique geodesic curve between μ_0 and μ_1 in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$. We are interested in the question to know whether μ_t is transport-regular for $t < 1$. Because of Corollary 1.3, for $\mu_0 \otimes \mu_0$-almost every x, x' we have

$$\langle \overrightarrow{xx'}, \overrightarrow{T_t x T_t x'} \rangle \geq (1-t)|x' - x|^2,$$

so that

$$|T_t x' - T_t x| \geq (1-t)|x' - x|.$$

Hence there exists a μ_t-measurable map S_t such that μ_0-almost surely $S_t \circ T_t = \mathrm{Id}$. Therefore $S_{t\#}\mu_t = \mu_0$ and

$$|\overrightarrow{y'y}| \geq \frac{1}{(1-t)}|\overrightarrow{S_t y' S_t y}|$$

for $\mu_t \otimes \mu_t$ almost every (y, y'). Then up to a modification of μ_t-measure 0, S_t is Lipschitz continuous on $\mathrm{Spt}(\mu_t)$. Thus we can consider that S_t is Lipschitz continuous and extend its definition domain to $\mathbb{R}^d$.

2.1. **Some particular cases.** Let $t \in]0,1[$ and $F \subset \mathbb{R}^d$ satisfying one of the properties of Example 1.9: $\mathcal{H}^d(F) = 0$, $\mathcal{H}^{d-1}(F) = 0$, $\dim \mathcal{H}(F) \leq d - 1$ or "F is a $d - 1$ dimensional Lipschitz piece". Consider $S_t(F)$ and observe that it satisfies the same property as F, just because S_t is Lipschitz continuous. Thus if a transport-regular μ_0 satisfies properties described in Example 1.9, μ_t satisfies the

same properties and it is transport-regular. On the contrary, in Example 1.10 the Lipschitz image of a $\mathcal{C}^2$ submanifold is not necessary $\mathcal{C}^2$. This fact is at the origin of the proof of Theorem 2.2.

2.2. **A counterexample in the general case.** The counterexample will take place in $\mathbb{R}^2$ and we will consider $t = 1/2$. Generalizations to higher dimension and other interpolation parameters will be given in Theorem 2.5. We start with a result of one variable analysis.

Proposition 2.1. *There exists a $\mathcal{C}^1$ function f defined on $[0, 1]$ such that for all $H \in \mathcal{C}^2([0, 1])$,*

$$(2.1) \qquad \mathcal{L}^1\left(\{x \in [0, 1],\ f(x) = H(x)\}\right) = 0.$$

See [3, Appendix] for statements with Hölder conditions.

Proof. It is enough to prove that some continuous function g meets the C^1 functions on a set of measure 0. Actually if there exists a continuous g satisfying

$$(2.2) \qquad \mathcal{L}^1(\{g = h\}) = 0$$

for all $h \in C^1([0, 1])$, we can set $f(x) = \int_0^x g$. Therefore for H of class $\mathcal{C}^2$, at density points of $\{f = H\}$ we must have $f' = H'$ so that $\mathcal{L}^1(\{f = H\}) \leq \mathcal{L}^1(\{g = H'\}) = 0$. One can see that a Weierstrass-like function (take for instance van der Waerden function) or almost surely a one-dimensional Brownian path can serve as examples of g. Actually these functions are nowhere *approximatively differentiable* on $[0, 1]$. See [8] and the references therein for those and other results. $\qquad\square$

Figure 1 represents the construction involved in the next (main) theorem.

Theorem 2.2. *There exists μ_0, μ_1, transport-regular measures of $\mathcal{P}_2(\mathbb{R}^2)$ such that $\mu_{1/2}$ is not transport-regular.*

Proof. Consider f as in Proposition 2.1. Up to multiplication by a constant one can assume that $|f'| < 1$. We introduce $q^+ : x \rightarrow (x, f(x))$ from $[0, 1]$ to the graph of f, $q^- : x \rightarrow (x, -f(x))$ and $p : (x, y) \rightarrow x$. We set now $\mu_0 = q^+_{\#}\nu$ and $\mu_1 = q^-_{\#}\nu$ where ν denotes the restriction of the Lebesgue measure to $[0, 1]$. Following part 1.2.2 we denote by $\vec{u}$ the unit vector of the x-axis. As $|f'| < 1$, $\preceq$ is a total order on spt μ_0 and spt μ_1. It follows from Lemma 1.6 and the comment after it that $\pi = (q^+ \otimes q^-)_{\#}\nu$ is the unique optimal transport plan between μ_0 and μ_1. But $\frac{q^+ + q^-}{2}(x) = (x, 0)$ so that $\mu_{1/2} = \mathcal{L}^1_{|[0,1] \times \{0\}}$ is concentrated on the x-axis. Thus $\mu_{1/2}$ is not transport-regular (consider $\phi(x, y) = |y|$ in Proposition 1.7 or compare with Example 4.9 in [12]).

The last thing, we have to check is that μ_0 and μ_1 are transport-regular. Of course it suffices to consider μ_0. Note that for any measurable set $F \subset \mathbb{R}^2$ we have

$$\mu_0(F) = \mathcal{L}^1\left(p(\mathrm{Spt}(\mu_0) \cap F)\right),$$

where $\mathrm{Spt}(\mu_0)$ is simply the graph of f in $\mathbb{R}^2$. Let ϕ be a convex function and S the set of the non-differentiability points of ϕ. We have to show $\mu_0(S) = 0$. As Alberti proved (Proposition 1.8), there are 1-dimensional submanifolds of class $\mathcal{C}^2$, $M_k \subset \mathbb{R}^d$ and a set N with $\mathcal{H}^1(N) = 0$ such that $S \subset \cup_{k=1}^{+\infty} M_k \cup N$. But

$$(2.3) \qquad \mu_0(N) = \mathcal{L}^1\left(p(\mathrm{Spt}(\mu_0) \cap N)\right) \leq \mathcal{H}^1(p(N)) \leq \mathcal{H}^1(N) = 0$$

because p is 1-Lipschitz. Let us prove now that $\mu_0(M_k) = 0$ for a given $k \in \mathbb{N}$. Because of Sard lemma, $\mathcal{H}^1(p(M_k^u)) = 0$ where $M_k^u = \{m \in M_k \mid \vec{u} \text{ is normal to } M_k\}$. Therefore

$$\mu_0(M_k^u) = \mathcal{L}^1\left(p(\mathrm{Spt}(\mu_0) \cap M_k^u)\right) = 0.$$

Moreover $M_k \setminus M_k^u$ is the countable union of pieces of graphs of class $\mathcal{C}^2$. Thus

$$\mathcal{L}^1\left(p(\mathrm{Spt}(\mu_0) \cap (M_k \setminus M_k^u))\right) = 0$$

just because f follows the consequence of Proposition 2.1. Finally $\mu_0(M_k) = 0$ and $\mu_0(S) = 0$ as we want. $\qquad\square$

Remark 2.3. We have actually proven that μ_0 satisfies the hypotheses of Example 1.10.

Remark 2.4. The optimal transport map T used in this construction is an orthogonal symmetry. This is a very special feature because for most of measures usually considered in optimal transport theory, pushing forward by symmetry provides a transport plan that does not satisfy the cyclical monotonicity.

2.2.1. *Generalization.* In this last part we generalize Theorem 2.2 to higher dimensions d and interpolation parameters $t \in]0,1[$. The proof for $d > 2$ can is deduced from $d = 2$ thanks to a tensorization method.

Theorem 2.5. *For any $d \geq 2$ and any $t \in]0,1[$, there exists μ_0, μ_1 transport-regular measures of $\mathcal{P}_2(\mathbb{R}^d)$ such that μ_t is not transport-regular.*

In a first time, we keep $t = 1/2$. Let's define

$$(2.4) \qquad \mu_i^{(d)} = \underbrace{\nu \otimes \cdots \otimes \nu}_{d-2} \otimes \mu_i$$

for $i = 0, 1$, where ν is the uniform measure on $[0,1]$ and μ_i is a measure of $\mathbb{R}^2$ as in Theorem 2.2. We claim that these measures are transport-regular. We show it for $\mu_0^{(d)}$. Observe that $\mu_0^{(d)} = q_\#^{+(d)} \nu^{\otimes d-1}$ where

$$q^{+(d)}(x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{d-1}, f(x_{d-1})),$$

for the same $f : \mathbb{R} \to \mathbb{R}^2$ as used in Theorem 2.2. Actually $\mu_0^{(d)}$ satisfies the hypotheses of Example 1.10. On the one hand $\mu_0^{(d)}$ is absolutely continuous with respect to $\mathcal{H}^{d-1}$. To show this, just consider a relation corresponding to (2.3) where $p^{(d)}(x_1, \ldots x_d) = (x_1, \ldots, x_{d-1})$ replaces p. On the other hand, any hypersurface of class $\mathcal{C}^2$ can be decomposed in two parts. The first part M' is made of the points where the the normal vector is in $\mathbb{R}^{d-1} \times \{0\}$. Then because of Sard lemma $\mathcal{L}^{d-1}(p^{(d)}(M')) = 0$. The other part is a countable union of graph pieces of class $\mathcal{C}^2$. Let M'' be one of these pieces. Hence

$$M'' = \{(x, v(x)) \mid x \in U\}$$

where $v \in \mathcal{C}^2(U, \mathbb{R})$ and $U \subset \mathbb{R}^{d-1}$ is an open set. As $v(x_1, \ldots, x_{d-2}, \cdot) \in \mathcal{C}^2(\mathbb{R})$ we can conclude using the proof of Theorem 2.2 and Fubini's theorem that

$$\mathcal{L}^{d-1}\left(p^{(d)}(\mathrm{Spt}(\mu_0^{(d)}) \cap M'')\right) = 0.$$

Thus μ_0 is transport-regular.

We prove now that formula (2.4) also holds for $i \in]0,1[$, i.e., the displacement interpolation commutes with the $\nu^{\otimes d-2}$-tensorization. From there we will be done

because $\mu_{1/2}^{(d)}$ is concentrated on $\mathbb{R}^{d-1} \times \{0\}$ so that it is not transport-regular. Consider the optimal transport plan $\pi^{(d)}$ between $\mu_0^{(d)}$ and $\mu_1^{(d)}$. For r defined by $r(x_1, \ldots, x_d) = (x_{d-1}, x_d)$ we obtain a coupling $\pi_r = (r \otimes r)_\# \pi^{(d)}$ between μ_0 and μ_1. This projection reduced the value of the cost in (1.1), we have $C(\pi_r) \le C(\pi^{(d)})$. But $C(\pi^{(d)}) \le C(((\mathrm{Id} \otimes \mathrm{Id})_\# \nu)^{\otimes d-2} \otimes \pi) = C(\pi)$ where π is the optimal transport plan between μ_0 and μ_1. Hence $\pi_r = \pi$. From there it is not yet difficult to see that

$$\pi^{(d)} = ((\mathrm{Id} \otimes \mathrm{Id})_\# \nu)^{\otimes d-2} \otimes \pi.$$

Note that in this proof, we have been implicitly identifying $\mathbb{R}^d \times \mathbb{R}^d$ with $(\mathbb{R} \times \mathbb{R})^d$.

For the generalization in the convexity parameter t, observe that for any $T > 1/2$, the non transport-regular $\mu_{1/2}$ is the interpolated measure of parameter $t = \frac{1}{2T} > 1/2$ between μ_0 and μ_T. Moreover μ_T is transport-regular because $\mu_T = u_\#^T \mu_1$ where $u^T : (x_1, \ldots, x_d) \to (x_1, \ldots, x_{d-1}, 2(T - \frac{1}{2})x_d)$ is affine and invertible. With a symmetric argument one can conclude for $t < 1/2$ too.

Remark 2.6. In the one dimensional case, transport-regular means with no atom. If μ_0 and μ_1 have no atom, it is the same for any transport plan π. For an optimal π, it follows from paragraph 1.2.1 that $x < x' \Rightarrow y \le y'$ for $(x, y), (x', y') \in \mathrm{Spt}(\pi)$. Hence for $t \in]0, 1[$, e_t is one-to-one when restricted to $\mathrm{Spt}(\pi)$, and μ_t has no atom.

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