# RECONCILIATION OF PROBABILITY MEASURES

JEAN BÉRARD AND NICOLAS JUILLET

Abstract. We discuss the reconciliation problem between probability measures: given $n \geqslant 2$ probability spaces $(\Omega, \mathcal{F}_1, \mathbb{P}_1), \ldots, (\Omega, \mathcal{F}_n, \mathbb{P}_n)$ with a common sample space, does there exist an overall probability measure $\mathbb{P}$ on $\mathcal{F} = \sigma(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ such that, for all $i$, the restriction of $\mathbb{P}$ to $\mathcal{F}_i$ coincides with $\mathbb{P}_i$ ? General criteria for the existence of a reconciliation are stated, along with some counterexamples that highlight some delicate issues. Connections to earlier (recent and far less recent) work are discussed, and elementary self-contained proofs for the various results are given.

## 1. Introduction

Consider a finite number $n \geqslant 2$ of probability spaces all built upon the same sample space $\Omega$, and denoted by $(\Omega, \mathcal{F}_1, \mathbb{P}_1), \ldots, (\Omega, \mathcal{F}_n, \mathbb{P}_n)$. We ask whether it is possible to *reconcile* these $n$ probability spaces, meaning that there exists a probability measure $\mathbb{P}$ on $\mathcal{F} = \sigma(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ such that, for all $1 \leqslant i \leqslant n$, the restriction of $\mathbb{P}$ to $\mathcal{F}_i$ coincides with $\mathbb{P}_i$. In such a case, we say that $\mathbb{P}$ provides a *reconciliation* of the probability measures $\mathbb{P}_1, \ldots, \mathbb{P}_n$.

This is a natural problem from a modeling perspective, where several probabilistic models may be available, each describing a specific aspect of the situation under study. The question is then the existence of a probabilistic model which simultaneously incorporates the previous specific models into a global one. Scenario aggregation (see e.g. [4]) and coherent belief modeling (see e.g. [1]) are two examples where related (though not equivalent) problems appear.

Note that, in general, the $\sigma-$fields $\mathcal{F}_1, \ldots, \mathcal{F}_n$ correspond to distinct but not completely unrelated families of events, which leads to additional constraints on a potential reconciliation $\mathbb{P}$ beyond the mere requirement that $\mathbb{P}_{|\mathcal{F}_i} = \mathbb{P}_i$ for all $i$. For instance, one may have two events $A_i \in \mathcal{F}_i$ and $A_j \in \mathcal{F}_j$ for which $A_i \cap A_j = \emptyset$ with $i \neq j$, so that any potential reconciliation $\mathbb{P}$ should satisfy $\mathbb{P}(A_i \cap A_j) = \emptyset$. As a consequence, even in the simple case where all $\sigma-$fields are assumed to be finite, the existence of a reconciliation $\mathbb{P}$ is neither automatic nor a trivial question.

In this paper, we first state a characterization of the existence of a reconciliation in the case of finite $\sigma-$fields. For $n = 2$, a very simple criterion is obtained, while the corresponding criterion in the general case $n \geqslant 2$ looks more complicated. Through a counterexample, we show that as simple a criterion as in the case $n = 2$ cannot be expected to hold in general. We then discuss the extension of these results to the case of more general (infinite) $\sigma-$fields. Through a counterexample, we show that additional conditions are required for such an extension to hold, then give one example of such conditions.

Note that the positive results stated in this paper can in fact be derived as corollaries of results obtained some decades ago by various authors, as we gradually became aware while completing the present study. The reconciliation problem as formulated above is neither the main motivation nor a notable example in these works, and we believe that the short self-contained proofs provided in the present paper are still of interest. Moreover, the counterexamples we provide highlight some interesting issues that are specific to the reconciliation problem.

The rest of the paper is organized as follows. Part 1.1 is devoted to the statement of the results (positive and negative). Connections with earlier work are discussed in Part 1.2. Finally, proofs of the various results are collected in Section 2.

## 1.1. **Statement of results.**

1.1.1. *The finite case.* Throughout this section, the $\sigma-$fields $\mathcal{F}_1, \dots, \mathcal{F}_n$ on $\Omega$ are assumed to comprise a finite number of events. Extensions to the infinite case are discussed in the next section.

The first result deals with the case of two probability spaces ($n = 2$), where we have the following characterization of when a reconciliation exists.

**Theorem 1.1.** *Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ comprise a finite number of events. Then it is possible to reconcile $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$ if and only if*

$$\mathbb{P}_1(E_1) \leqslant \mathbb{P}_2(E_2) \text{ for every } E_1 \subset E_2 \text{ with } E_1 \in \mathcal{F}_1 \text{ and } E_2 \in \mathcal{F}_2. \tag{1}$$

Note that condition (1) is clearly necessary since, given a reconciliation $\mathbb{P}$, one must have $\mathbb{P}_1(E_1) = \mathbb{P}(E_1) \leqslant \mathbb{P}(E_2) \leqslant \mathbb{P}_2(E_2)$ as soon as $E_1 \subset E_2$. The non-trivial part of the theorem lies in the fact that condition (1) is indeed sufficient to ensure the existence of a reconciliation. Also note that, taking complementary sets, condition (1) is equivalent to the symmetric condition $E_2 \subset E_1 \Rightarrow \mathbb{P}_2(E_2) \leqslant \mathbb{P}_1(E_1)$.

The next result deals with the general case $n \geqslant 2$. The existence of a reconciliation is characterized in terms of elementary integer-valued measurable functions. Consider integer numbers $m_1 \geqslant 1, \dots, m_n \geqslant 1$, and, for all $1 \leqslant i \leqslant n$, a family of $m_i$ pairwise disjoint events $E_1^{(i)} \in \mathcal{F}_i, \dots, E_{m_i}^{(i)} \in \mathcal{F}_i$, and a family of $m_i$ integer numbers $d_1^{(i)} \in \mathbb{Z}, \dots, d_{m_i}^{(i)} \in \mathbb{Z}$. Then set

$$f_i = \sum_{\ell=1}^{m_i} d_\ell^{(i)} \cdot \mathbf{1}_{E_\ell^{(i)}} \text{ and } f = \sum_{i=1}^{n} f_i. \tag{2}$$

Note that each $f_i$ is $\mathcal{F}_i-$measurable, and that one has

$$\int f_i d\mathbb{P}_i = \sum_{\ell=1}^{m_i} d_\ell^{(i)} \cdot \mathbb{P}_i(E_\ell^{(i)}). \tag{3}$$

**Theorem 1.2.** *Assume that $\mathcal{F}_1, \dots, \mathcal{F}_n$ comprise a finite number of events. Then it is possible to reconcile $(\Omega, \mathcal{F}_1, \mathbb{P}_1), \dots, (\Omega, \mathcal{F}_n, \mathbb{P}_n)$ if and only if, for all functions $f$ and $f_1, \dots, f_n$ of the form given by* (2)*, one has that*

$$f \geqslant 0 \Rightarrow \sum_{i=1}^{n} \int f_i d\mathbb{P}_i \geqslant 0. \tag{4}$$

The necessity of condition (4) above is easy to see, for, given a reconciliation $\mathbb{P}$, and a non-negative function $f$, one must have

$$0 \leqslant \int f d\mathbb{P} = \sum_{i=1}^{n} \int f_i d\mathbb{P} = \sum_{i=1}^{n} \int f_i d\mathbb{P}_i.$$

As a consequence, the non-trivial part of Theorem 1.2 is the fact that condition (4) is indeed sufficient for the existence of a reconciliation $\mathbb{P}$.

Note that we insisted on giving a formulation in terms of integer-valued functions (instead of general real-valued functions) since we are interested in having a combinatorial interpretation of the criterion in terms of comparisons between probabilities of sets. Indeed, condition (4) in Theorem 1.1 has a very clear such combinatorial interpretation, and one may hope for an equally clear criterion in the general case $n \geqslant 2$. Since, in the case $n = 2$, the existence of a reconciliation can be checked by looking at a very specific subset of the conditions appearing in Theorem 1.2 (namely, those involving $m_1 = 1$, $m_2 = 1$, $d_1^{(1)} = -1$, $d_1^{(2)} = 1$), a natural question is whether, in the general case $n \geqslant 2$, it is still possible to characterize the existence of a reconciliation through a subset of conditions that involve only "small" integer values. Unfortunately, the following result shows that the answer is negative.

**Theorem 1.3.** *For all $K > 0$, there exists a triple of probability spaces $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$, $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$, $(\Omega, \mathcal{F}_3, \mathbb{P}_3)$, where $\mathcal{F}_i$ comprises a finite number of events for $i = 1, 2, 3$, such that:*

- *no reconciliation exists;*
- *condition* (4) *is satisfied whenever $|d_\ell^{(i)}| \leqslant K$ for all $i, \ell$.*

Theorem 1.3 shows that, even when $n = 3$, there is no upper bound on how large the integer coefficients $d_\ell^{(i)}$ in (2) have to be in order to check the existence of a reconciliation. Note that, with $K = 1$ (and Theorem 1.1), the theorem also provides an example of a triple of probability spaces among which every pair admits a reconciliation, while no overall reconciliation exists for the triple.

1.1.2. *Extension to the infinite case.* We now consider the case where the $\sigma$−fields $\mathcal{F}_i$ on $\Omega$ may comprise an infinite number of events. We start with a negative result showing that one cannot extend the previous results to such a general case without additional assumptions.

**Theorem 1.4.** *There exists a pair* $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$, $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$ *such that condition* (1) *is satisfied, but for which no reconciliation exists.*

The next theorem states that, under a (reasonably mild and general) additional assumption on the $\sigma$−fields $\mathcal{F}_1, \dots, \mathcal{F}_n$, can indeed be extended.

**Theorem 1.5.** *Assume that the $\sigma$−fields $\mathcal{F}_1, \dots, \mathcal{F}_n$ (with $n = 2$ in the case of Theorem 1.1) are of the form $\mathcal{F}_i = \sigma(X_i)$, where $X_i$ is a map from $\Omega$ to $\mathbb{R}^{d_i}$ (equipped with the Borel $\sigma$−field), with $d_i \geqslant 1$. Moreover, assume that the set $(X_1, \dots, X_n)(\Omega)$ is a closed set in $\mathbb{R}^{d_1+\cdots+d_n}$. Then the conclusions of Theorems 1.1 and 1.2 hold. Namely, in the case $n = 2$, there exists a reconciliation between $\mathbb{P}_1$ and $\mathbb{P}_2$ if and only if condition* (1) *holds, and, in the general case $n \geqslant 2$, there exists a reconciliation between $\mathbb{P}_1, \dots, \mathbb{P}_n$ if and only if condition* (4) *holds.*

The two following corollaries are immediate by-products of the theorem.

**Corollary 1.6.** *The conclusions of Theorems 1.1 and 1.2 hold when the $\sigma$−fields $\mathcal{F}_1, \dots, \mathcal{F}_n$ are generated by a countable number of atoms.*

**Corollary 1.7.** *Let $V$ be a closed set of $\mathbb{R}^d \times \mathbb{R}^d$ and $\mu$, $\nu$ two probability measures on $\mathbb{R}^d$. Then, using transport terminology, there exists a* transport plan *$\pi$ from $\mu$ to $\nu$ concentrated on $V$, i.e, a measure $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals $\mu$ and $\nu$ and $\pi(V) = 1$, if and only if for every Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$*

$$\mu(A) \leqslant \nu(B) \text{ as soon as } (A \times \mathbb{R}^d) \cap V \subset (\mathbb{R}^d \times B) \cap V. \tag{5}$$

1.2. **Connections with earlier work.** Early references bearing directly on Theorem 1.1 are Fréchet [7] and Dall'Aglio [5] (where an unpublished result of Berge is also quoted). There, a general result is proved on the existence of two-dimensional discrete distributions with prescribed marginals and upper bounds, from which Theorem 1.1 can be derived. Subsequent work by Kellerer (Satz 3.2 in [9]) in a general framework (finite dimensional distributions on abstract spaces) can be used to derive Theorem 1.2. Finally, key ingredients needed to prove Theorem 1.5 can be taken from Theorem 11 in Strassen [14]. We refer to the book [6] for additional references and a more detailed historical perspective on this body of work.

Here, a step-by-step self-contained approach to the proofs is given. Indeed, in our specific framework, Theorem 1.2 stems from a rather straightforward application of Farkas' Lemma. Theorem 1.1 is then derived from Theorem 1.2 through a simple reduction argument. Finally, Theorem 1.5 is what one more or less readily obtains by passing to the limit within a sequence of discrete approximations provided by Theorems 1.1 and 1.2.

To conclude this section, we would like to emphasize the connections of Corollary 1.7 (which is in fact a variant of Theorem 11 in [14], and also related with ) with recent developments in optimal transport theory.

Although it appears a little less obvious, condition (5) is necessary in Corollary 1.7 for the same reason why (1) is in Theorem 1.1. Its natural interpretation is not probabilistic, but in terms of transport, as follows. We recall that $\pi(A \times B)$ represents the mass transported from $A$ to $B$. The constraint on the capacity of the transport plan is given by $V$: no mass can travel from $x$ to $y$ if $(x, y) \in V$. Gravel located in $A$ with mass $\mu(A)$ can be displaced according to the capacity transport constraint encoded by $V$ only to the set $B_0 = \mathrm{proj}^2((A \times \mathbb{R}) \cap V)$. The storage size of this (universally measurable) set is $\nu(B_0)$. In order for the transport to be manageable, it has to be larger than $\mu(A)$, which is condition (5). Corollary 1.7 states that under this condition there exists a transport plan that satisfies the capacity constraint. As for the usual Monge–Kantorovich optimal transport problem, the goal in constrained versions of the problem is to obtain information on the minimizers of $\pi \mapsto \iint c \, \mathrm{d}\pi$, where $c : \mathbb{R}^d \times \mathbb{R}^d$ is a lower semicontinuous cost function. In particular, is there a Monge solution, that is in the form $\pi = (\mathrm{Id} \otimes T)_{\#}\mu$ ?

We are aware of two works in the transport literature that correspond to this problem. In [8] the special case of a constraint displacement vector has been encoded by $V = \{(x, y) \in \mathbb{R}^{2d} : y - x \in \overrightarrow{V}\}$ where $\overrightarrow{V} \subset \mathbb{R}^d$ is a closed convex set with some additional properties. The authors look at the shape of optimizers for a quadratic (constraint) cost $c(x, y) = |y - x|^2 1_{y-x\in\overrightarrow{V}} + \infty \cdot 1_{y-x\notin\overrightarrow{V}}$. They prove that, provided $\mu$ is absolutely continuous, any solution $\pi^*$ of the Monge–Kantorovich transport problem is a Monge transport plan $\pi^* = (\mathrm{Id} \otimes T)_{\#}\mu$ where $T(x) \in x + \overrightarrow{V}$, $\mu$−almost surely, and hence, $\pi^*$ is uniquely determined. This occurs under the assumption made that a transport plan $\pi$ with finite cost does exist, i.e, $\iint c \, \mathrm{d}\pi < \infty$ for some admissible $\pi$. In this respect, condition (5) is namely required in order for the problem to have finite minimal total cost. Note that (5) reads in this case $\nu(A + \overrightarrow{V}) \geqslant \mu(A)$. The other work is [2] where $\overrightarrow{V}$ is the unit Euclidean ball $\mathcal{B}_1(0)$ and $c$ is the so-called relativistic cost $c(x, y) = (1 - \sqrt{1 - |y - x|^2})1_{|y-x|\leqslant 1} + \infty \cdot 1_{|y-x|>1}$. As $c$ is bounded on its domain, (5) is a necessary and sufficient condition to have finite total cost. The authors also introduce $c_t := c(x/t, y/t)$ and the critical speed $T = T_{\mu,\nu} = \inf\{t \in \mathbb{R}_+ : \text{the total cost is finite for } c_t\}$. Therefore $T$ is also the minimal $t$ such that $\nu(A + \mathcal{B}_t(0)) \geqslant \mu(A)$ for every Borel set $A$. Let us finally mention [12, 11] where another Monge-Kantorovich problem under capacity constraint is investigated: a transport plan $\pi$ is admissible if it possesses a density $h$ on $X \times Y \subset \mathbb{R}^{d_1+d_2}$ that satisfies $0 < h \leqslant \bar{h}$ for a given $\bar{h}$. In this case the authors of [12] cite (p. 575) two functional criteria for the existence of an admissible transport plan. These are due to Kellerer [9] and Levin [13]. An equivalent set-based criterion due to Kellerer [10, Satz

4.2] directly transposes in the continuous settings the (already mentioned) ones obtained by Fréchet [7] and Dall'Aglio [5] in the discrete setting.

## 2. Proofs

We start with the proof of Theorem 1.2, which consists merely in an application of Farkas' Lemma.

*Proof of Theorem 1.2.* As already noted after the statement of the theorem, condition (4) is clearly necessary for the existence of a reconciliation. We now assume that (4) holds, and prove the existence of a reconciliation.

Denote by $H$ the (finite-dimensional) vector space formed by real-valued functions $f$ on $\Omega$ of the form $f = f_1 + \cdots + f_n$, where each $f_i$ is an $\mathcal{F}_i-$measurable real-valued function on $\Omega$. Note that condition (4) is assumed to hold for such functions $f$ when each $f_i$ is integer-valued. Since we consider functions on a finite state space, condition (4) in fact holds for any $f$ in $H$, as can be seen by approximating each $f_i$ by functions with rational values, which in turn can be written as integer values divided by the g.c.d. of these values.

Now, for any $\omega \in \Omega$, denote by $\varphi_\omega$ the linear form on $H$ defined by $\varphi_\omega(f) = f(\omega)$. On the other hand, denote by $\theta$ the linear form on $H$ defined by $\theta(f) = \sum_{i=1}^n \int f d\mathbb{P}_i$. Condition (4) (extended to all functions in $H$) says that, if $f \in H$ is such that $\varphi_\omega(f) \geqslant 0$ for all $\omega \in \Omega$, then one has $\theta(f) \geqslant 0$. Farkas' Lemma then guarantees the existence of a family $(t_\omega)_{\omega\in\Omega}$ of *non-negative* real numbers such that, for all $f \in H$, one has the identity

$$\theta(f) = \sum_{\omega\in\Omega} t_\omega \cdot f(\omega). \tag{6}$$

Applying (6) with e.g. $f_1 \equiv 1$ and $f_2 = \cdots = f_n \equiv 0$, we see that the non-negative numbers $(t_\omega)_{\omega\in\Omega}$ sum up to 1, so that we can define $\mathbb{P}(\{\omega\}) = t_\omega$. Given $A_i \in \mathcal{F}_i$, we can apply (6) with $f_i = \mathbf{1}_{A_i}$ and $f_j \equiv 0$ for $j \neq i$, and we deduce that $\mathbb{P}(A_i) = \mathbb{P}_i(A_i)$. □

We now prove Theorem 1.1 from Theorem 1.2, through a suitable reduction argument.

*Proof of Theorem 1.1.* As already noted after the statement of the theorem, condition (1) is clearly necessary for the existence of a reconciliation. We now assume that (1) holds, and prove the existence of a reconciliation through Theorem 1.2.

Without loss of generality, we start with a function $f$ of the form $f = f_1 + f_2$, with $f_1 = \sum_i \lambda_i \mathbf{1}_{A_i}$ and $f_2 = -\sum_j \mu_j \mathbf{1}_{B_j}$, where the $\lambda_i$ and $\mu_j$ are in $\mathbb{Z}$, and where the $(A_i)_i$ (resp. $(B_j)_j$) is a finite family of events in $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) that forms a finite partition of $\Omega$. In the sequel, the $\lambda_i$ and $\mu_j$ are called the coefficients of $f$.

Our goal is to establish condition (4) so as to apply Theorem 1.2. So we now have to prove that, if $f \geqslant 0$, then $\sum_i \lambda_i \mathbb{P}_1(A_i) - \sum_j \mu_j \mathbb{P}_2(B_j) \geqslant 0$.

*Step 1: We may assume that $f$ has coefficients in $\mathbb{N}$:* Adding to $f$ the function $0 = C - C = C(\sum_i \mathbf{1}_{A_i}) - C(\sum_j \mathbf{1}_{B_j})$ where $C = \max_{i,j}(|\lambda_i|, |\mu_j|)$, we see that

$$\sum_i (\lambda_i + C)\mathbf{1}_{A_i} - \sum_j (\mu_j + C)\mathbf{1}_{B_j}$$

has non-negative coefficients $\lambda_i' = \lambda_i + C$ and $\mu_j' = \mu_j + C$, while keeping $\sum \lambda_i'\mathbb{P}_1(A_i) + \sum \mu_j'\mathbb{P}_2(B_j) = \sum \lambda_i\mathbb{P}_1(A_i) + \sum \mu_j\mathbb{P}_2(B_j)$, since $\sum C\mathbb{P}_1(A_i) - \sum C\mathbb{P}_2(B_j) = 0$.

*Step 2: We may assume that $f$ has coefficients in $\{0,1\}$:* Consider $f$ such that $f \geqslant 0$, and, given Step 1, assume that the coefficients in $f$ satisfy $\lambda_i, \mu_j \in \mathbb{N}$. We set $\lambda_i' = \max(\lambda_i - 1, 0)$ and $\mu_j' = \max(\mu_j - 1, 0)$, and write $f = g + h$, with $g = \sum \lambda_i'\mathbf{1}_{A_i} - \sum \mu_j'\mathbf{1}_{B_j}$ and $h = \sum(\lambda_i - \lambda_i')\mathbf{1}_{A_i} - \sum(\mu_j - \mu_j')\mathbf{1}_{B_j}$. Since $\sum \lambda_i\mathbf{1}_{A_i} \leqslant \sum \mu_j\mathbf{1}_{B_j}$, we must have, for every $(i,j)$ with $A_i \cap B_j \neq \emptyset$, the fact that $\lambda_i \leqslant \mu_j$, and therefore that $\lambda_i' \leqslant \mu_j'$ and $\lambda_i - \lambda_i' \leqslant \mu_j - \mu_j'$. As a consequence, we have $g \geqslant 0$ and $h \geqslant 0$, and $g$ and $h$ have non-negative integer coefficients. Moreover, unless they are already equal to 0 or 1, the coefficients of $g$ and $h$ are strictly smaller than the corresponding coefficients of $f$. Indeed, if $\lambda_i \geqslant 2$, one has $\lambda_i' < \lambda_i$ and $\lambda_i - \lambda_i' < \lambda_i$, and similarly for $\mu_j$. As a consequence, after a finite number of iterations, we can write $f$ as a sum of non-negative functions with integer coefficients in $\{0,1\}$.

*Conclusion:* For a function $f \geqslant 0$ with coefficients in $\{0,1\}$, the difference $\sum_i \lambda_i\mathbb{P}_1(A_i) - \sum_j \mu_j\mathbb{P}_2(B_j)$ is of the form $\mathbb{P}_1(E_1) - \mathbb{P}_2(E_2)$ with $E_2 \subset E_1$, so that condition (1) implies the fact that $\sum_i \lambda_i\mathbb{P}_1(A_i) - \sum_j \mu_j\mathbb{P}_2(B_j) \geqslant 0$. □

We now prove Theorem 1.3, which consists in building an example with no reconciliation, but for which any exception to condition (4) must involve at least one large coefficient.

*Proof of Theorem 1.3.* Given an integer number $n \geqslant 2$, we define a sample space $\Omega$ with $3n$ distinct elements $\Omega = \{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n\}$. The $\sigma$ field $\mathcal{F}_1$ contains $A_1 = \{a_1, c_1\}, \dots, A_n = \{a_n, c_n\}$ and $A_0 = \{b_1, \dots, b_n\}$. The $\sigma$ field $\mathcal{F}_2$ contains $B_1 = \{b_1, c_1\}, \dots, B_n = \{b_n, c_n\}$ and $B_0 = \{a_1, \dots, a_n\}$. Finally, the $\sigma-$field $\mathcal{F}_3$ contains $C_1 = \{a_1, b_1\}$, $C_{n+1} = \{c_n\}$, and for every $k \in \{2, \dots, n\}$ the set $C_k = \{a_k, b_k, c_{k-1}\}$. For $i = 1, 2, 3$, the probability measure $\mathbb{P}_i$ gives the same mass to each of the $n+1$ events in $\mathcal{F}_i$, namely $1/n + 1$. See Figure 2 for a visual presentation.

*There exists no reconciliation between $\mathbb{P}_1$,$\mathbb{P}_2$ and $\mathbb{P}_3$:* Observe that, by construction, we have the inequality:

$$\sum_{k=1}^{n+1} 2^{k-1}\mathbf{1}_{C_k} \leqslant \sum_{k=1}^{n} 2^{k-1}(\mathbf{1}_{A_k} + \mathbf{1}_{B_k}).$$

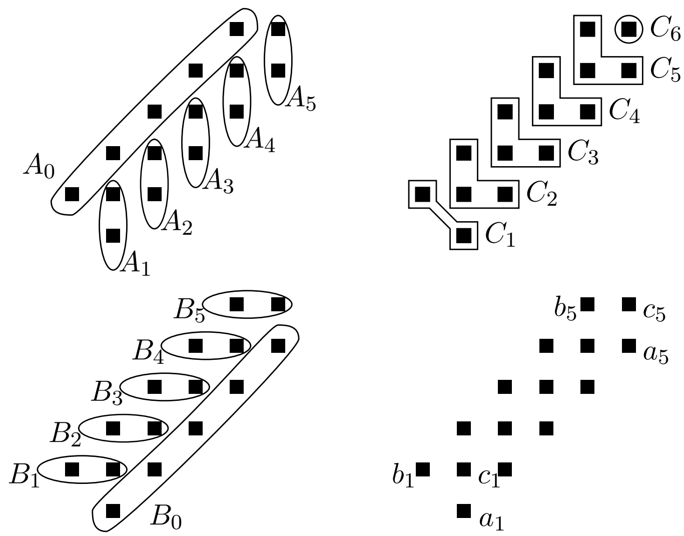


FIGURE 1. The set $\Omega$ with $(\mathcal{F}_i)_{i\in\{1,2,3\}}$.

As a consequence, if a reconciliation $\mathbb{P}$ existed, taking the expectation with respect to $\mathbb{P}$ in the above equation, we would have

$$\frac{1}{n+1}\times\left(\sum_{k=0}^{n}2^k\right)\leqslant\frac{1}{n+1}\times\left(2\sum_{k=0}^{n-1}2^k\right),$$

a contradiction.

*Any exception to condition* (4) *has at least one large coefficient:* consider an exception to condition (4) of the following form: $(\lambda_i)_{0\leqslant i\leqslant n}$, $(\mu_j)_{0\leqslant j\leqslant n}$ and $(\nu_k)_{1\leqslant k\leqslant n+1}$ are integer coefficients such that

$$\sum\lambda_i\mathbf{1}_{A_i}+\sum\mu_j\mathbf{1}_{B_j}\geqslant\sum\nu_k\mathbf{1}_{C_k}, \tag{7}$$

but

$$\sum\lambda_i\left(\frac{1}{n+1}\right)+\sum\mu_j\left(\frac{1}{n+1}\right)<\sum\nu_k\left(\frac{1}{n+1}\right). \tag{8}$$

Focusing on (7) at points $c_1,c_2,\ldots,c_n$ we see that $\nu_{k+1}\leqslant\lambda_k+\mu_k$ for $k=1,\ldots,n$. Therefore

$$\sum_{k=1}^{n}\lambda_k+\mu_k\geqslant\sum_{k=1}^{n}\nu_{k+1}.$$

In view of (8), the remaining coefficients $\lambda_0$, $\mu_0$ and $\nu_1$ have to satisfy the inequality $\lambda_0+\mu_0<\nu_1$, and, moreover, for every $k=1,\ldots,n$, we must have $\lambda_k+\mu_k-\nu_{k+1}<\delta:=\nu_1-(\lambda_0+\mu_0)$. On the other hand, from (7) at points $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ we find that $\nu_k\leqslant\lambda_k+\mu_0$ and $\nu_k\leqslant\mu_k+\lambda_0$.

Now, for every $k \in \{1, \dots, n\}$ we find that

$$\begin{aligned}\nu_{k+1} &> \lambda_k + \mu_k - \delta \\ &\geqslant (\nu_k - \lambda_0) + (\nu_k - \mu_0) - \delta \\ &\geqslant 2\nu_k - \nu_1,\end{aligned}$$

from which we deduce that

$$\nu_{k+1} - \nu_1 > 2(\nu_k - \nu_1). \tag{9}$$

Using the fact that $\nu_2$ is an integer, we first deduce that $\nu_2 \geqslant \nu_1 + 1$. Then, iterating the inequality, we obtain that $\nu_{n+1} - \nu_1 \geqslant 2^{n-1}$. As a consequence, we must have either $|\nu_1| \geqslant 2^{n-2}$ or $|\nu_{n+1}| \geqslant 2^{n-2}$. The conclusion of the theorem is thus proved, by choosing an $n$ such that $2^{n-2} > K$. □

We now prove Theorem 1.4, which shows that Theorem 1.1 does not hold in the general (infinite) case, at least without additional assumptions.

*Proof of Theorem 1.4.* Take $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_1 < x_2 < 1\}$, and let $X_1$ and $X_2$ denote the coordinate maps on $\Omega$ (so that $X_i(x_1, x_2) = x_i$ for $i = 1, 2$). Then define the $\sigma-$fields $\mathcal{F}_i = \sigma(X_i)$ for $i = 1, 2$. Finally, let $\mathbb{P}_i$ be the probability on $(\Omega, \mathcal{F}_i)$ uniquely characterized by the fact that $X_i$ follows the uniform distribution on $]0, 1[$.

We first check that $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$ satisfy condition (1). To this end, consider $E_1 \in \mathcal{F}_1$ and $E_2 \in \mathcal{F}_2$ such that $E_1 \subset E_2$. By definition of $\mathcal{F}_1$, there exists a Borel set $B_1 \subset ]0, 1[$ such that $E_1 = \{(x_1, x_2);\ x_1 \in B_1, x_2 \in ]x_1, 1[\}$. Similarly, there exists a Borel set $B_2 \subset ]0, 1[$ such that $E_2 = \{(x_1, x_2);\ x_2 \in B_2, x_1 \in ]0, x_2[\}$. From the fact that $E_1 \subset E_2$, one deduces that $B_2$ must contain the set $]0, \sup B_1[$. Since $X_1$ and $X_2$ both follow the uniform distribution on $]0, 1[$, we deduce that $\mathbb{P}_1(E_1) = \mathbb{P}_1(X_1 \in B_1) \leqslant \mathbb{P}_1(X_1 \in ]0, \sup B_1]) = \mathbb{P}_2(X_2 \in ]0, \sup B_1]) = \mathbb{P}_2(X_2 \in ]0, \sup B_1[) \leqslant \mathbb{P}_2(E_2)$. We conclude that $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$ satisfy condition (1).

Let us know prove by contradiction that there exists no reconciliation between $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$. If $\mathbb{P}$ where such a reconciliation, both $X_1$ and $X_2$ would follow the uniform distribution on $]0, 1[$ when viewed as random variables on $(\Omega, \sigma(\mathcal{F}_1, \mathcal{F}_2), \mathbb{P})$, so that we would have:

$$\int_\Omega X_1(\omega) d\mathbb{P}(\omega) = \int_\Omega X_2(\omega) d\mathbb{P}(\omega) = 1/2. \tag{10}$$

On the other hand, one has $X_1 \leqslant X_2$ by definition of $\Omega$, so that (10) implies that $\mathbb{P}(X_1 = X_2) = 1$. Since, by definition of $\Omega$ again, one has $X_1(\omega) < X_2(\omega)$ for all $\omega \in \Omega$, we would also have $\mathbb{P}(X_1 = X_2) = 0$, a contradiction. □

We now prove Theorem 1.5, which gives one possible extension of Theorems 1.1 and 1.2 beyond the finite discrete case. The proof strategy simply consists in taking a limit in a sequence of discrete approximations whose existence is provided by the previous results.

*Proof of Theorem 1.5.* We give the proof for Theorem 1.1 only, the proof for Theorem 1.2 being competely similar. Let $V = (X_1, X_2)(\Omega)$, and assume that, within the assumptions of the theorem, $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfy (1). For $i = 1, 2$, let $\widetilde{\mathbb{P}}_i$ denote the probability distribution of $X_i$ viewed as a real-valued random variable on $(\Omega, \mathcal{F}_i, \mathbb{P}_i)$. Denote also $\widetilde{X}_1$ and $\widetilde{X}_2$ the canonical coordinate maps on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

We first claim that, to prove the theorem, it is enough to prove that there exists a probability measure $\widetilde{\mathbb{P}}$ on $\mathbb{R}^{d_1+d_2}$ (equipped with the Borel $\sigma-$field) such that :

- $\widetilde{\mathbb{P}}(V) = 1$;
- for $i = 1, 2$, $\widetilde{\mathbb{P}}(\widetilde{X}_i \in B) = \widetilde{\mathbb{P}}_i(B)$ for every Borel set $B \subset \mathbb{R}^{d_i}$.

Indeed, consider such a probability measure $\widetilde{\mathbb{P}}$. For any $D \in \sigma(X_1, X_2)$, there exists a Borel set $C \subset V$ such that $D = \{(X_1, X_2) \in C\}$. We thus define $\mathbb{P}$ on $\mathcal{F} = \sigma(X_1, X_2)$ by letting $\mathbb{P}(\{(X_1, X_2) \in C\}) = \widetilde{\mathbb{P}}(C)$, for every Borel subset $C \subset V$. Since $V = (X_1, X_2)(\Omega)$, this leads to a well-defined probability measure on $\mathcal{F}$. Moreover, given $D_i \in \sigma(X_i)$, one can write $D_i = \{X_i \in C_i\}$, with $C_i$ a Borel set in $\mathbb{R}^{d_i}$, and one then has $\mathbb{P}(D_i) = \mathbb{P}(X_i \in C_i) = \widetilde{\mathbb{P}}(\{\widetilde{X}_i \in C_i\} \cap V) = \widetilde{\mathbb{P}}(\widetilde{X}_i \in C_i) = \widetilde{\mathbb{P}}_i(C_i) = \mathbb{P}_i(X_i \in C_i) = \mathbb{P}_i(D_i)$, so that $\mathbb{P}$ provides a reconciliation between $\mathbb{P}_1$ and $\mathbb{P}_2$.

We now establish the existence of the required probability measure $\widetilde{\mathbb{P}}$. Let us consider a sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise distinct real numbers, such that $\{a_n; n \in \mathbb{N}\}$ is a dense subset in $\mathbb{R}$. Given $n \geqslant 0$, denote by $b_n^{(1)} < \cdots < b_n^{(n)}$ the ordered $n-$tuple obtained by ordering the values $a_1, \ldots, a_n$. Then define the intervals $I_n^{(0)} = ]-\infty, b_n^{(1)}]$, $I_n^{(k)} = ]b_n^{(k)}, b_n^{(k+1)}]$ for $1 \leqslant k \leqslant n-1$, and $I_n^{(n)} = ]b_n^{(n)}, +\infty[$. Given a $d-$tuple $\ell = (\ell_1, \ldots, \ell_d) \in \{0, \ldots, n\}^d$, let $I_n^{(\ell)} = I_n^{(\ell_1)} \times \cdots \times I_n^{(\ell_d)}$. Given such a a $d_1-$tuple $k_1$ and a $d_2-$tuple $k_2$, we denote by $(k_1, k_2)$ the $(d_1 + d_2)-$tuple obtained by concatenating $k_1$ and $k_2$.

Let now $K_n$ denote the subset formed by the pairs $(k_1, k_2) \in \{0, \ldots, n\}^{d_1} \times \{0, \ldots, n\}^{d_2}$ for which $I_n^{(k_1,k_2)} \cap V \neq \emptyset$. Finally, for $k \in \{0, \ldots, n\}^{d_i}$, let $p_{i,n}^{(k)} = \mathbb{P}_i(X_i \in I_n^{(k_i)})$.

Assumption (1) on $\mathbb{P}_1, \mathbb{P}_2$ and Theorem 1.1 show the existence of a discrete probability $(p_n^{(k_1,k_2)})_{(k_1,k_2)\in K_n}$ on the set $K_n \subset \{0, \ldots, n\}^{d_1} \times \{0, \ldots, n\}^{d_2}$ whose marginals are $(p_{1,n}^{(k)})_{k\in\{0,\ldots,n\}^{d_1}}$ and $(p_{2,n}^{(k)})_{k\in\{0,\ldots,n\}^{d_1}}$ respectively. Now, for $(k_1, k_2) \in K$, let us denote by $x_n^{(k_1,k_2)}$ an arbitrary element in $I_n^{(k_1,k_2)} \cap V$, and define a probability measure on $\mathbb{R}^{d_1+d_2}$ by

$$\widetilde{\mathbb{P}}^n = \sum_{(k_1,k_2)\in K_n} p_n^{(k_1,k_2)} \cdot \delta_{x_n^{(k_1,k_2)}}.$$

We note that $\widetilde{\mathbb{P}}^n(V) = 1$, and that, for $i = 1, 2$, and any $k \in \{0, \ldots, n\}^{d_i}$, one has that $\widetilde{\mathbb{P}}^n(\widetilde{X}_i \in I_n^{(k)}) = p_{i,n}^{(k)} = \mathbb{P}_i(X_i \in I_n^{(k)}) = \widetilde{\mathbb{P}}_i(I_n^{(k)})$. Moreover, by construction, for all $m \geqslant n$, every set $I_m^{(k)}$, with $k \in \{0, \ldots, n\}^{d_i}$, is a disjoint (finite) union of sets of the form $I_n^{(\ell)}$, with $\ell \in \{0,\dots,n\}^{d_i}$, so that $\widetilde{\mathbb{P}}^m(\widetilde{X}_i \in I_n^{(k)}) = \widetilde{\mathbb{P}}^n(\widetilde{X}_i \in I_n^{(k)}) = \widetilde{\mathbb{P}}_i(I_n^{(k)})$.

Let us now check that the sequence of probability measures $(\widetilde{\mathbb{P}}^n)_{n\geqslant 0}$ is tight. By density, we must have that $\lim_{n\to+\infty} b_n^{(1)} = -\infty$ and $\lim_{n\to+\infty} b_n^{(n)} = +\infty$. As a consequence, for $i = 1,2$, and $1 \leqslant j \leqslant d_i$, one has that $\lim_{n\to+\infty} \widetilde{\mathbb{P}}_i\left(\varpi_j^{-1}(]-\infty, b_n^{(1)}])\right) = 0$ and $\lim_{n\to+\infty} \widetilde{\mathbb{P}}_i\left(\varpi_j^{-1}(]b_n^{(n)}, +\infty[)\right) = 0$, where $\varpi_j : \mathbb{R}^{d_i} \to \mathbb{R}$ denotes the projection onto the $j-$th coordinate.

Moreover, the sets $\varpi_j^{-1}(]-\infty, b_n^{(1)}])$ and $\varpi_j^{-1}(]b_n^{(n)}, +\infty[)$ can be written as disjoint (finite) unions of sets of the form $I_n^{(\ell)}$, with $\ell \in \{0,\dots,n\}^{d_i}$, so that

$$\begin{aligned}\widetilde{\mathbb{P}}^m\left(\varpi_j(\widetilde{X}_i) \in ]-\infty, b_n^{(1)}] \cup ]b_n^{(n)}, +\infty[)\right) &= \widetilde{\mathbb{P}}^n\left(\varpi_j(\widetilde{X}_i) \in ]-\infty, b_n^{(1)}] \cup ]b_n^{(n)}, +\infty[\right) \\ &= \widetilde{\mathbb{P}}_i\left(\varpi_j^{-1}(]-\infty, b_n^{(1)}] \cup ]b_n^{(n)}, +\infty[)\right).\end{aligned}$$

We deduce that, for $i = 1,2$, and $1 \leqslant j \leqslant d_i$, one has

$$\lim_{n\to+\infty} \sup_{m\geqslant n} \widetilde{\mathbb{P}}^m\left(\varpi_j(\widetilde{X}_i) \in ]-\infty, b_n^{(1)}] \cup ]b_n^{(n)}, +\infty[\right) = 0,$$

which is enough to establish the tightness of the sequence $(\widetilde{\mathbb{P}}^n)_{n\geqslant 0}$.

We now invoke Prohorov's theorem to deduce the existence of a subsequence $(\widetilde{\mathbb{P}}^{n_r})_{r\geqslant 0}$ which converges in distribution to a probability $\widetilde{\mathbb{P}}$ on $\mathbb{R}^2$.

Since $\widetilde{\mathbb{P}}^n(V) = 1$ for all $n$ and $V$ is a closed set, we deduce that $\widetilde{\mathbb{P}}(V) = 1$. To conclude the proof, it remains to prove that $\widetilde{\mathbb{P}}(\widetilde{X}_i \in B) = \widetilde{\mathbb{P}}_i(B)$ for every Borel set $B \subset \mathbb{R}$. First note that, for $i = 1,2$, the sequence of probability distributions $\widetilde{\mathbb{P}}^{n_r}(\widetilde{X}_i \in \cdot)$ converges weakly to the limit $\widetilde{\mathbb{P}}(\widetilde{X}_i \in \cdot)$. On the other hand, for any $n \geqslant 0$ and $k \in \{0,\dots,n\}^{d_i}$, we have, for all $m \geqslant n$, the identity $\widetilde{\mathbb{P}}^m(\widetilde{X}_i \in I_n^{(k)}) = \widetilde{\mathbb{P}}_i(I_n^{(k)})$. As an immediate consequence, for any $n \geqslant 0$, we have that $\lim_{r\to+\infty} \widetilde{\mathbb{P}}^{n_r}(\widetilde{X}_i \in I_n^{(k)}) = \widetilde{\mathbb{P}}_i(I_n^{(k)})$. Moreover, by density, every open set in $\mathbb{R}^{d_i}$ can be written as a finite or countable union of sets of the form $I_n^{(k)}$, and the intersection of a finite number of such sets is still of the same form. By [3, Theorem 2.2, page 17], we deduce that the sequence of probability distributions $\widetilde{\mathbb{P}}^{n_r}(\widetilde{X}_i \in \cdot)$ converges weakly to the limit $\widetilde{\mathbb{P}}_i(\cdot)$. We deduce that the two weak limits $\widetilde{\mathbb{P}}(\widetilde{X}_i \in \cdot)$ and $\widetilde{\mathbb{P}}_i(\cdot)$ are identical. □

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Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67 000 Strasbourg, France

*E-mail address*: `jean.berard@math.unistra.fr, nicolas.juillet@math.unistra.fr`