

# Instability of martingale optimal transport in dimension $d \geq 2^*$

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## Abstract

Stability of the value function and the set of minimizers w.r.t. the given data is a desirable feature of optimal transport problems. For the classical Kantorovich transport problem, stability is satisfied under mild assumptions and in general frameworks such as the one of Polish spaces. However, for the martingale transport problem several works based on different strategies established stability results for  $\mathbb{R}$  only. We show that the restriction to dimension  $d = 1$  is not accidental by presenting a sequence of marginal distributions on  $\mathbb{R}^2$  for which the martingale optimal transport problem is neither stable w.r.t. the value nor the set of minimizers. Our construction adapts to any dimension  $d \geq 2$ . For  $d \geq 2$  it also provides a contradiction to the martingale Wasserstein inequality established by Jourdain and Margheriti in  $d = 1$ .

**Keywords:** martingale optimal transport; stability; convex order; martingale couplings; Monge-Kantorovich transport problems.

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## 1 Introduction

For two probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}^d$  let  $\Pi(\mu, \nu)$  denote the set of all couplings between  $\mu$  and  $\nu$ , i.e. the set of all probability measures on  $\mathbb{R}^d \times \mathbb{R}^d$  which have marginal distributions  $\mu$  and  $\nu$ . Let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be measurable and integrable with respect to the elements of  $\Pi(\mu, \nu)$ . The classical optimal transport problem is given by

$$V_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\pi(x, y). \quad (\text{OT})$$

For the cost function  $c_1(x, y) := \|y - x\|$  (where  $\|\cdot\|$  is the Euclidean norm) the 1-Wasserstein distance  $\mathcal{W}_1 := V_{c_1}$  is a metric on  $\mathcal{P}_1(\mathbb{R}^d)$ , the space of probability measures  $\mu$  that satisfy  $\int_{\mathbb{R}^d} \|x\| d\mu(x) < \infty$ .

Two probability measures  $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$  are said to be in convex order, denoted by  $\mu \leq_c \nu$ , if  $\int_{\mathbb{R}^d} \varphi d\mu \leq \int_{\mathbb{R}^d} \varphi d\nu$  for all convex functions  $\varphi \in L^1(\nu)$ . If  $\mu \leq_c \nu$ , Strassen's

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theorem yields that there exists at least one martingale coupling between  $\mu$  and  $\nu$ . A martingale coupling is a coupling  $\pi \in \Pi(\mu, \nu)$  for which there exists a disintegration  $(\pi_x)_{x \in \mathbb{R}^d}$  such that

$$\int_{\mathbb{R}} y \, d\pi_x(y) = x \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.1}$$

If  $\mu \leq_c \nu$ , the martingale optimal transport problem is given by

$$V_c^M(\mu, \nu) = \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \, d\pi(x, y) \tag{MOT}$$

where  $\Pi_M(\mu, \nu)$  denotes the set of all martingale couplings between  $\mu$  and  $\nu$ .

**Stability in  $d = 1$**

Let us recall two reasons why stability results are crucial from an applied perspective. Firstly, they enable the strategy of approximating the problem by a discretized problem or by any other that can rapidly be solved computationally (cf. [1, 14]). Secondly, any application to noisy data would require stability for the results to be meaningful. In relation with (MOT), this discussion is motivated by its connection to (robust) mathematical finance (cf. [4, 12]).

Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  with  $\mu \leq_c \nu$ , and  $(\mu_n)_{n \in \mathbb{N}}$  and  $(\nu_n)_{n \in \mathbb{N}}$  be sequences of probability measures on  $\mathbb{R}$  with finite first moment such that  $\lim_{n \rightarrow \infty} \mathcal{W}_1(\mu_n, \mu) = 0$ ,  $\lim_{n \rightarrow \infty} \mathcal{W}_1(\nu_n, \nu) = 0$  and  $\mu_n \leq_c \nu_n$  for all  $n \in \mathbb{N}$ . The following stability results are available:

- (S1) *Accumulation Points of Minimizers:* Let  $c$  be a continuous cost function which is sufficiently integrable (e.g.  $|c(x, y)| \leq A(1 + |x| + |y|)$ ) and let  $\pi_n$  be a minimizer of the (MOT) problem between  $\mu_n$  and  $\nu_n$  for all  $n \in \mathbb{N}$ . Any weak accumulation point of  $(\pi_n)_{n \in \mathbb{N}}$  is a minimizer of (MOT) between  $\mu$  and  $\nu$ .
- (S2) *Continuity of the Value:* Let  $c$  be a continuous cost function which is sufficiently integrable (e.g.  $|c(x, y)| \leq A(1 + |x| + |y|)$ ). There holds

$$\lim_{n \rightarrow \infty} V_c^M(\mu_n, \nu_n) = V_c^M(\mu, \nu).$$

- (S3) *Approximation:* For all  $\pi \in \Pi_M(\mu, \nu)$  there exists a sequence  $(\pi_n)_{n \in \mathbb{N}}$  of martingale couplings between  $\mu_n$  and  $\nu_n$  that converges weakly to  $\pi$ .

This constitutes the heart of the theory of stability recently consolidated for the martingale transport problem on the real line. Before we go more into the details of the literature let us stress that with (S3) any minimizer can be approximated by a sequence of martingale transport with prescribed marginals. Therefore, under mild assumptions (S3) implies (S2). Moreover, due to the tightness of  $\bigcup_{n \in \mathbb{N}} \Pi_M(\mu_n, \nu_n)$ , (S2) implies (S1).

Early versions of (S1) and (S2) for special classes of cost-functions were obtained by Juillet [16] and later by Guo and Obloj [11]. The general version of (S1) and (S2) was first shown by Backhoff-Veraguas and Pammer [2, Theorem 1.1, Corollary 1.2] and Wiesel [22, Theorem 2.9]. Only very recently, Beiglböck, Jourdain, Margheriti and Pammer [3] have proven (S3). We want to stress that (S1), (S2) and (S3) are given in a minimal formulation and that in the articles some aspects of the results are notably stronger. For instance, the cost function  $c$  in (S1) and (S2) can be replaced by a uniformly converging sequence  $(c_n)_{n \in \mathbb{N}}$  [2]. Moreover, it is an important achievement that on top of weak convergence we have convergence w.r.t. (an extension of) the adapted Wasserstein metric for the approximation in (S3) [3] and for the convergence in (S1) [6], see also [22]. Finally, these stability results also hold for weak martingale optimal transport which is an extension of (MOT) w.r.t. the structure of the cost function (cf. [6, Theorem 2.6]). For further details we invite the interested reader to directly consult the articles.

The martingale Wasserstein inequality introduced by Jourdain and Margheriti in [13, Theorem 2.12] belongs also to the context of stability and approximation and it appears for example as the important last step in the proof of (S3) in [3]. In dimension  $d = 1$  there exists a constant  $C > 0$  independent of  $\mu$  and  $\nu$  such that

$$\mathcal{M}_1(\mu, \nu) \leq C\mathcal{W}_1(\mu, \nu). \tag{MWI}$$

where  $\mathcal{M}_1(\mu, \nu)$  is the value of the (MOT) problem between  $\mu$  and  $\nu$  w.r.t. the cost function  $\|x - y\|$ . Moreover, they proved that  $C = 2$  is sharp. For their proof Jourdain and Margheriti introduce a family of martingale couplings  $\pi \in \Pi_M(\mu, \nu)$  that satisfy  $\int_{\mathbb{R} \times \mathbb{R}} |x - y| d\pi(x, y) \leq 2\mathcal{W}_1(\mu, \nu)$  (including the particularly notable inverse transform martingale coupling).

**Instability in  $d \geq 2$**

The stability of (OT) (and its extension to weak optimal transport [6, Theorem 2.5]) is independent of the dimension. However, Beiglböck *et al.* had to restrict their stability theorem for (weak) (MOT) in the critical step to dimension  $d = 1$  (cf. [6, Theorem 2.6 (b’)]). Similarly, in dimension  $d \geq 2$ , Jourdain and Margheriti could only extend the martingale Wasserstein inequality for product measures and for measures in relation through a homothetic transformation, see [13, Section 3]. The difficulties in expanding these stability results to higher dimensions are not of technical nature but a consequence of instability of (MOT) in higher dimensions without further assumptions.

In the following we construct a sequence of probability measures on  $\mathbb{R}^2$  for which (S1), (S2) and (S3) do not hold. Moreover, we provide an example that shows that the inequality (MWI) does not hold in dimension  $d = 2$  for any fixed constant  $C > 0$  without further assumptions. Since one can embed this example into  $\mathbb{R}^d$  for any  $d \geq 3$  by the map  $\iota : (x, y) \mapsto (x, y, 0, \dots, 0)$ , these results also fail in any higher dimension.

We denote by  $P_\theta$  the one step probability kernel of the simple random walk along the line  $l_\theta$  that makes an angle  $\theta \in [0, \frac{\pi}{2}]$  with the  $x$ -axis. More precisely:

$$P_\theta : \mathbb{R}^2 \ni (x_1, x_2) \mapsto \frac{1}{2} (\delta_{(x_1, x_2) + (\cos(\theta), \sin(\theta))} + \delta_{(x_1, x_2) - (\cos(\theta), \sin(\theta))}) \in \mathcal{P}_1(\mathbb{R}^2).$$

For  $m, n \in \mathbb{N}_{\geq 1}$  we define two probability measures on  $\mathbb{R}^2$  by

$$\mu_m := \sum_{i=1}^m \frac{1}{m} \delta_{(i, 0)} \quad \text{and} \quad \nu_{m,n} := \mu_m P_{\frac{\pi}{2n}}$$

where  $\mu_m P_{\frac{\pi}{2n}}$  denotes the application of the kernel  $P_{\frac{\pi}{2n}}$  to  $\mu_m$ . Figure 1 illustrates  $(\mu_2, \nu_{2,2})$  and  $(\mu_3, \nu_{3,3})$ .

Since for any convex function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  Jensen’s inequality yields

$$\int_{\mathbb{R}^2} \varphi d\nu_{m,n} = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \varphi dP_{\frac{\pi}{2n}}(x, \cdot) \right) d\mu_m(x) \geq \int_{\mathbb{R}^2} \varphi d\mu_m,$$

we have  $\mu_m \leq_c \nu_{m,n}$  for all  $m, n \in \mathbb{N}_{\geq 1}$ .

**Lemma 1.1.** *The martingale coupling  $\pi_{m,n} := \mu_m(\text{Id}, P_{\frac{\pi}{2n}})$  is the only martingale coupling between  $\mu_m$  and  $\nu_{m,n}$  for all  $m, n \in \mathbb{N}_{\geq 1}$ . (Here  $\mu_m(\text{Id}, P_{\frac{\pi}{2n}})$  denotes the application of the kernel  $(\text{Id}, P_{\frac{\pi}{2n}})$  to  $\mu_m$ .)*

For every  $m \geq 2$  the sequence  $(\mu_m, \nu_{m,n})_{n \in \mathbb{N}}$  serves as a counterexample to analogue versions of (S1), (S2) and (S3) in dimension  $d = 2$ . The crucial observation is that whereas  $\Pi_M(\mu_m, \nu_{m,n})$  consists of exactly one element for all  $n \in \mathbb{N}$  by Lemma 1.1, there are infinitely many different martingale couplings between  $\mu_m$  and the limit of  $(\nu_{m,n})_{n \in \mathbb{N}}$ .

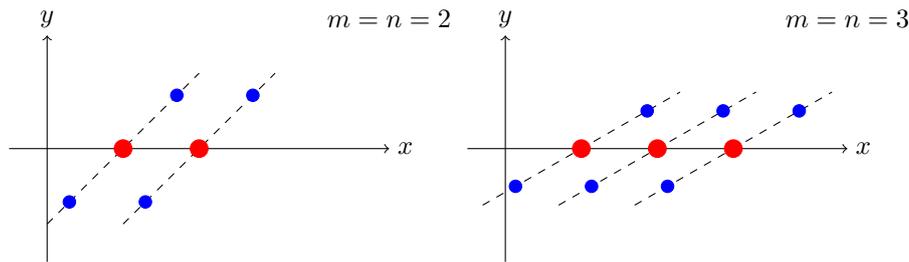


Figure 1: The construction for  $m = n = 2$  and  $m = n = 3$ . The red circles indicate the Dirac measures of  $\mu_m$  each with mass  $\frac{1}{m}$  and the blue circles indicate the Dirac measures of  $\nu_{m,n}$  each with mass  $\frac{1}{2m}$ .

**Proposition 1.2.** *Let  $m \geq 2$ . There holds  $\lim_{n \rightarrow \infty} \mathcal{W}_1(\nu_{m,n}, \mu_m P_0) = 0$ .*

*Moreover, we have the following:*

- (i) *Let  $c_1(x, y) := \|y - x\|$  for all  $x, y \in \mathbb{R}^2$  and  $\pi_{m,n} := \mu_m(\text{Id}, P_{\frac{\pi}{2n}})$  for all  $n \in \mathbb{N}_{\geq 1}$ . The martingale couplings  $\pi_{m,n}$  are minimizers of the (MOT) problem between  $\mu_m$  and  $\nu_{m,n}$  w.r.t.  $c_1$ . Moreover,  $(\pi_{m,n})_{n \in \mathbb{N}}$  is weakly convergent but the limit is not an optimizer of (MOT) w.r.t.  $c_1$  between (its marginals)  $\mu_m$  and  $\mu_m P_0$ .*
- (ii) *Let  $c_1$  be defined as in (i). There holds*

$$\lim_{n \rightarrow \infty} V_{c_1}^M(\mu_m, \nu_{m,n}) = 1 > V_{c_1}^M(\mu_m, \mu_m P_0).$$

- (iii) *The set  $\Pi_M(\mu_m, \mu_m P_0) \setminus \{\mu_m(\text{Id}, P_0)\}$  is non empty and no element in this set can be approximated by a weakly convergent sequence  $(\pi_{m,n})_{n \in \mathbb{N}}$  of martingale couplings  $\pi_{m,n} \in \Pi_M(\mu_m, \nu_{m,n})$ .*

The sequence  $(\mu_n, \nu_{n,n})_{n \in \mathbb{N}}$  shows that there cannot exist a constant  $C > 0$  for which the inequality (MWI) holds in dimension  $d = 2$ .

**Proposition 1.3.** *There holds*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{M}_1(\mu_n, \nu_{n,n})}{\mathcal{W}_1(\mu_n, \nu_{n,n})} = +\infty.$$

**Remark 1.4.** The theory of MOT in dimension two and further is also challenging in other aspects. For instance, the concept of irreducible components and convex paving can be directly reduced to the study of potential functions in dimension  $d = 1$ , whereas there are at least three different advanced approaches in dimension  $d \geq 2$  (cf. [10, 8, 20]). On the level of processes we would like to remind the reader that a higher dimensional version of Kellerer’s theorem [18, Theorem 3] is still not proved or disproved. One major obstacle is that the one-dimensional proof via Lipschitz-Markov kernels cannot be extrapolated, see [15, Section 2.2] where a method similar to ours is used.

## 2 Proofs

We denote by  $f_{\#}\mu$  the push-forward of the measure  $\mu$  under the function  $f$ .

*Proof of Lemma 1.1.* <sup>1</sup> Let  $m, n \geq 2$  be integers and  $\theta_n := \frac{\pi}{2n}$ . We denote by  $L_{\theta_n}$  the projection parallel to the line  $l_{\theta_n} = \{(x_1, \tan(\theta_n)x_1) : x_1 \in \mathbb{R}\}$  onto the  $x$ -axis, i.e.

$$L_{\theta_n} : \mathbb{R}^2 \ni (x_1, x_2) \mapsto x_1 - \tan(\theta_n)^{-1}x_2 \in \mathbb{R}.$$

<sup>1</sup>As suggested by the referee an alternative simple proof is a proof by induction starting from right-most atom of  $\mu_m$ . However, this argument can not be adapted to the situation of Remark 3.2.

Moreover, by setting  $\tilde{\nu} := (L_{\theta_n})_{\#}\nu_{m,n}$  and  $\tilde{\mu} := (L_{\theta_n})_{\#}\mu_m$  one has

$$\tilde{\nu} = \frac{1}{m} \sum_{i=1}^m \delta_i = \tilde{\mu}. \tag{2.1}$$

Let  $\pi \in \Pi_M(\mu_m, \nu_{m,n})$ . As  $L_{\theta_n}$  is a linear map,  $\tilde{\pi} := (L_{\theta_n} \otimes L_{\theta_n})_{\#}\pi$  is a martingale coupling of  $\tilde{\mu}$  and  $\tilde{\nu}$ . Indeed, for all  $\varphi \in C_b(\mathbb{R}^2)$  there holds

$$\int_{\mathbb{R}^2} \varphi(x)(y-x) d\tilde{\pi}(x) = L_{\theta_n} \left( \int_{\mathbb{R}^2} \varphi(L_{\theta_n}(x))(y-x) d\pi(x) \right) = 0$$

and this property is equivalent to  $\tilde{\pi}$  being a martingale coupling. Jensen's inequality in conjunction with (2.1) yields

$$\int_{\mathbb{R}} x^2 d\tilde{\mu}(x) \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} y^2 d\tilde{\pi}_x(y) \right) d\tilde{\mu}(x) = \int_{\mathbb{R}} y^2 d\tilde{\nu}(y) = \int_{\mathbb{R}} x^2 d\tilde{\mu}(x)$$

where  $(\tilde{\pi}_x)_{x \in \mathbb{R}}$  is a disintegration of  $\tilde{\pi}$  that satisfies (1.1). Since the square is a strictly convex function, there holds  $\int_{\mathbb{R}} y^2 d\tilde{\pi}_x = x^2$  if and only if  $\tilde{\pi}_x = \delta_x$ . Thus,  $\tilde{\pi} = \tilde{\mu}(\text{Id}, \text{Id})$  and we obtain

$$x_1 = L_{\theta_n}(x_1, x_2) = L_{\theta_n}(y_1, y_2) \quad \text{for } \pi\text{-a.e. } ((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$$

because  $\text{supp}(\mu_m) \subset \mathbb{R} \times \{0\}$ . Hence, the martingale transport plan  $\pi$  is only transporting along the lines parallel to  $l_{\theta_n}$ . Since there are exactly two points in the support of  $\nu_{m,n}$  that lie on the same line, and we are looking for a martingale coupling, we have

$$\pi = \mu_m(\text{Id}, P_{\theta_n}). \quad \square$$

**Lemma 2.1.** For all  $m \in \mathbb{N} \setminus \{0\}$  and  $\theta \in [0, \frac{\pi}{2}]$  one has

$$\mathcal{W}_1(\mu_m P_0, \mu_m P_{\theta}) \leq \theta.$$

*Proof.* The inequality consists merely of a comparison of angle and chord. Alternatively, for all  $m \in \mathbb{N}$  and  $\theta \in [0, \frac{\pi}{2}]$  we directly compute

$$\begin{aligned} \mathcal{W}_1(\mu_m P_0, \mu_m P_{\theta}) &\leq \int_{\mathbb{R}^2} \mathcal{W}_1(\delta_x P_0, \delta_x P_{\theta}) d\mu_m(x) \\ &= |(\sin(\theta), \cos(\theta) - 1)| = \sqrt{2(1 - \cos(\theta))} = 2 \sin(\theta/2) \leq \theta. \quad \square \end{aligned}$$

*Proof of Proposition 1.2.* Let  $m \geq 2$ . By Lemma 2.1, we know

$$\lim_{n \rightarrow \infty} \mathcal{W}_1(\nu_{m,n}, \mu_m P_0) = \lim_{n \rightarrow \infty} \mathcal{W}_1(\mu_m P_{\frac{\pi}{2n}}, \mu_m P_0) = 0.$$

Moreover, for all  $n \in \mathbb{N}$  Lemma 1.1 yields that  $\pi_{m,n} := \mu_m(\text{Id}, P_{\frac{\pi}{2n}})$  is the only martingale coupling between  $\mu_m$  and  $\nu_{m,n}$  and therefore automatically the unique minimizer of the (MOT) problem between these two marginals w.r.t. to any cost function. The sequence  $(\pi_{m,n})_{n \in \mathbb{N}}$  converges weakly to  $\pi_m := \mu_m(\text{Id}, P_0) \in \Pi_M(\mu_m, \mu_m P_0)$ . Note that

$$\begin{aligned} \pi'_m &:= \frac{1}{2m} \left( \delta_{((1,0),(1,0))} + 2 \sum_{i=2}^{m-1} \delta_{((i,0),(i,0))} + \delta_{((m,0),(m,0))} \right) \\ &\quad + \frac{1}{2m(m+1)} (m\delta_{((1,0),(0,0))} + \delta_{((1,0),(m+1,0))} + \delta_{((m,0),(0,0))} + m\delta_{((m,0),(m+1,0))}) \end{aligned}$$

is a martingale coupling between  $\mu_m$  and  $\mu_m P_0 = \mu_m - \frac{1}{m} \left( \frac{\delta_{(1,0)} + \delta_{(m,0)}}{2} - \frac{\delta_{(0,0)} + \delta_{(m+1,0)}}{2} \right)$  different from  $\pi_m$  where only the mass not shared by  $\mu_m$  and  $\mu_m P_0$  is moved.

Item (i): Since  $\pi_m$  is the weak limit of the sequence  $(\pi_{m,n})_{n \in \mathbb{N}}$ , it is the only accumulation point. But as we see below in (ii),  $\pi_m$  is not the minimizer of the (MOT) problem between  $\mu_m$  and  $\mu_m P_0$  w.r.t.  $c_1$ .

Item (ii): There holds

$$\begin{aligned} \lim_{n \rightarrow \infty} V_{c_1}^M(\mu_m, \nu_{m,n}) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \|x - y\| d\pi_{m,n} = 1 \\ &> \frac{2}{m+1} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \|x - y\| d\pi' \geq V_{c_1}^M(\mu_m, \mu_m P_0). \end{aligned}$$

In fact, according to Lim's result [19, Theorem 2.4], under an optimal martingale transport w.r.t.  $c_1$  the shared mass between the marginal distribution is not moving. Since  $\pi'_m$  is the unique martingale transport between  $\mu_m$  and  $\mu_m P_0$  with this property, it is the minimizer of this (MOT) problem and  $V_{c_1}^M(\mu_m, \mu_m P_0) = \frac{2}{m+1}$ .

Item (iii): Since  $\pi$  is the weak limit of the solitary elements of  $\Pi_M(\mu_m, \nu_{m,n})$ , no element of  $\Pi_M(\mu_m, \mu_m P_0) \setminus \{\mu_m(\text{Id}, P_0)\}$  can be approximated and  $\pi'_m$  is an element of this set. □

*Proof of Proposition 1.3.* Let  $n \in \mathbb{N}$ . By Lemma 1.1,  $\mu_n(\text{Id}, P_{\frac{\pi}{2n}})$  is the only martingale coupling between  $\mu_n$  and  $\nu_{n,n}$ . Thus,

$$\mathcal{M}_1(\mu_n, \nu_{n,n}) = 1.$$

Since  $\mathcal{W}_1$  is a metric on  $\mathcal{P}_1(\mathbb{R}^2)$ , the triangle inequality yields

$$\mathcal{W}_1(\mu_n, \nu_{n,n}) \leq \mathcal{W}_1(\mu_n, \mu_n P^0) + \mathcal{W}_1(\mu_n P^0, \nu_{n,n}).$$

We can easily compute

$$\mu_n P^0 = \frac{1}{2n} \left( \sum_{i=1}^n \delta_{(i-1,0)} + \sum_{i=1}^n \delta_{(i+1,0)} \right)$$

and therefore  $\mathcal{W}_1(\mu_n, \mu_n P^0) = \frac{1}{n}$ . By Lemma 2.1, there holds  $\mathcal{W}_1(\mu_n P^0, \nu_{n,n}) \leq \frac{\pi}{2n}$ . Hence, we obtain

$$\lim_{n \rightarrow \infty} \frac{\mathcal{M}_1(\mu_n, \nu_{n,n})}{\mathcal{W}_1(\mu_n, \nu_{n,n})} \geq \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n} + \frac{\pi}{2n}} = +\infty. \quad \square$$

### 3 Additional remarks

**Remark 3.1 (Duality).** Let  $m, n \geq 2$ . Recall that by their definitions the measures  $\mu_m$  and  $\nu_{m,n}$  are concentrated on the union of  $m$  lines parallel to  $l_{\theta_n}$  and going through  $(i, 0)$ ,  $i \in \{1, \dots, m\}$ . As in the proof of Lemma 1.1 we denote by  $L_{\theta_n}$  the linear map that maps the points of the  $i$ -th line to the integer  $i$ . We define three functions  $f_n, g_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $h_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$f_n(x) = (L_{\theta_n}(x))^2, \quad g_n(y) = 1 - (L_{\theta_n}(y))^2, \quad h_n(x) = 2L_{\theta_n}(x)w_n$$

where  $w_n \in \mathbb{R}^2$  represents  $L_{\theta_n}$  in the sense  $L_{\theta_n}(x) = \langle w_n, x \rangle$  for all  $x \in \mathbb{R}^2$ . For all  $x, y \in \mathbb{R}^2$  we have

$$\begin{aligned} f_n(x) + g_n(y) + \langle h_n(x), y - x \rangle &= (L_{\theta_n}(x))^2 + 1 - (L_{\theta_n}(y))^2 + 2L_{\theta_n}(x)(L_{\theta_n}(y) - L_{\theta_n}(x)) \\ &= 1 - (L_{\theta_n}(y) - L_{\theta_n}(x))^2. \end{aligned}$$

Since the measures  $\mu_m$  and  $\nu_{m,n}$  are concentrated on the union of the  $m$  parallel lines, for  $\mu_m \otimes \nu_{m,n}$ -a.e.  $(x, y) \in \mathbb{R}^2$  we have either  $(L_{\theta_n}(y) - L_{\theta_n}(x))^2 \geq 1$  if  $x$  and  $y$  are on different lines or  $|y - x| = 1$  if they are on the same line. Therefore,

$$f_n(x) + g_n(y) + \langle h_n(x), y - x \rangle \leq |y - x|, \quad \mu_m \otimes \nu_{m,n}\text{-a.e.} \tag{3.1}$$

Furthermore, we know

$$\begin{aligned} \int_{\mathbb{R}^2} f_n \, d\mu_m + \int_{\mathbb{R}^2} g \, d\nu_{m,n} &= 1 + \int_{\mathbb{R}^2} (L_{\theta_n}(x))^2 d\mu_m(x) - \int_{\mathbb{R}^2} (L_{\theta_n}(y))^2 d\nu_{m,n}(y) \\ &= 1 = V_{c_1}^M(\mu_m, \nu_{m,n}), \end{aligned} \tag{3.2}$$

i.e. the triple  $(f_n, g_n, h_n)$  is an optimizer to the martingale dual problem in the martingale sense [4, 10].<sup>2</sup> As a dual counterpart of the instability, this triple (that depends on  $n$ ) does not converge to an admissible triple as  $n$  goes to infinity. In fact, such a limit triple would impose a lower bound to the primal problem that is too large. Outside of the first axis the limit does not exist (in particular, it is impossible to project to the first axis with a map  $L_\theta$  where angle  $\theta$  is 0). On the first axis the pointwise limit<sup>3</sup> of  $(f_n, g_n, h_n)$  is given by  $(x_1^2, 1 - y_1^2, 2x_1)$ . This triple does not satisfy (3.1) for  $x = y$  whereas this latter is possible at the limit because the supports of  $\mu_m$  and  $\mu_m P_0$  collide.

A particular aspect of martingale transport duality is pointwise duality in the quasi-sure sense over the set of martingale transport plans as discovered in [5] (see also [10] and the preprint [7] for the multidimensional case). Since in our example this set is reduced to the single transport plan  $\pi_{m,n} := \mu_m(\text{Id}, P_{\frac{\pi}{2n}})$ , one can easily conclude with the existence of plenty of dual maximizers (still in the quasi-sure sense) for instance  $(f_n(x), g_n(y), h_n(x)) = (0, 1, (0, 0))$ , or  $(f_n(x), g_n(y), h_n(x)) = (f_n(x), f_n(x) + 1, (0, 0))$   $\pi_{m,n}$ -a.e. Again going to the limit does not provide  $\mu_m$  and  $\mu_m P_0$  with admissible triples since the supports of  $\nu_{m,n}$  and  $\mu_m$  collide.

**Remark 3.2 (Variations).** Our construction may appear somewhat degenerate since  $\mu_m$  is discrete and supported on a lower dimensional subspace of  $\mathbb{R}^2$ . Moreover the number of irreducible components (recall Remark 1.4 and the references therein) is not the same in the sequence and for the limit measures. However, it is not particularly difficult to adapt the present construction with new measures that appear more general but yield the same result:

- (i) One could replace the rows of Dirac measures by uniform measures on parallelograms. More precisely, we could set

$$\tilde{\mu}_{m,n} := \text{Unif}_{F_{m,n}} \quad \text{and} \quad \tilde{\nu}_{m,n} := \frac{1}{2} \left( \text{Unif}_{F_{m,n}^+} + \text{Unif}_{F_{m,n}^-} \right)$$

where  $F_m$  denotes the parallelogram spanned by the points

$$-v_n, \quad -v_n + (m, 0), \quad v_n + (m, 0) \quad \text{and} \quad v_n$$

with  $v_n := \frac{1}{3} \left( \cos\left(\frac{\pi}{2n}\right), \sin\left(\frac{\pi}{2n}\right) \right) \in \mathbb{R}^2$  and  $F_{m,n}^\pm$  is the translation of this parallelogram by  $\pm 3v_n$  (cf. Figure 2). By the same argument as in Lemma 1.1, any martingale coupling between  $\tilde{\mu}_{m,n}$  and  $\tilde{\nu}_{m,n}$  can only transport along lines parallel to  $\{(x, \tan\left(\frac{\pi}{2n}\right)x) : x \in \mathbb{R}\}$ . In contrast to the situation in Lemma 1.1, the

<sup>2</sup>To be completely rigorous with respect to the statement in the referred papers, it seems that Estimate (3.1) not only has to be satisfied almost everywhere but really for every  $(x, y) \in \mathbb{R}^2$ . This can be done by replacing  $f_n$  and  $g_n$  by functions  $\tilde{f}_n, \tilde{g}_n$  of the same equivalence class defined by  $\tilde{f}_n := f_n - 1$  and  $\tilde{g}_n(y) := g_n - 1$  outside the support of  $\mu_m$  and  $\nu_{m,n}$  respectively. Of course (3.2) is still true with the modified functions.

<sup>3</sup>In fact  $h_n$  does not converge, but interpreted as a linear form and restricted to the first axis we find the limit  $x_1 \mapsto 2x_1$ .

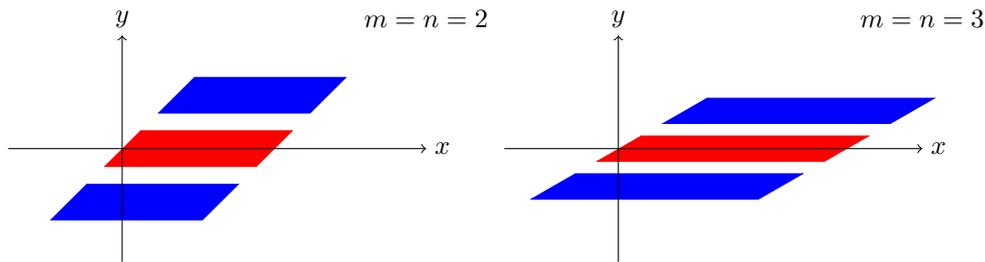


Figure 2: The construction in Remark 3.2 (i) for  $m = n = 2$  and  $m = n = 3$ . The red area is the support of  $\tilde{\mu}_{m,n}$  and the blue area the supports of  $\tilde{\nu}_{m,n}$ .

martingale transport along one of these parallel lines is no longer unique but every  $\pi \in \Pi_M(\tilde{\mu}_{m,n}, \tilde{\nu}_{m,n})$  satisfies  $\pi(|x - y| < \frac{1}{3}) = 0$  for all  $m, n \in \mathbb{N}$  because the supports are disjoint. This restriction carries over to any weak accumulation point of those martingale couplings and is sufficient to show analogous versions of Proposition 1.2 and Proposition 1.3.

- (ii) One could replace  $\mu_m$  and  $\nu_{m,n}$  by

$$\tilde{\mu}_m := (1 - \epsilon)\mu_m + \epsilon\gamma \quad \text{and} \quad \tilde{\nu}_{m,n} := (1 - \epsilon)\nu_{m,n} + \epsilon\gamma$$

where  $\epsilon \in (0, 1)$  and  $\gamma$  is a probability measure with full support (e.g. a standard normal distribution). There holds  $\mathcal{W}_1(\tilde{\mu}_m, \tilde{\nu}_{m,n}) = (1 - \epsilon)\mathcal{W}_1(\mu_m, \nu_{m,n})$  since  $\mathcal{W}_1$  derives of the Kantorovich-Rubinstein norm [17] (alternatively see [21, Bib. Notes of Ch.6 ] or [9, §\*11.8]) and  $\mathcal{M}_1(\tilde{\mu}_m, \tilde{\nu}_{m,n}) = (1 - \epsilon)\mathcal{M}_1(\mu_m, \nu_{m,n})$  by the result of Lim [19, Theorem 2.4].

- (iii) In order to show that our construction is not influenced by the number of irreducible components ( $m$  components in the sequence and 1 at the limit) let us introduce for  $m = 2$  the measures

$$\tilde{\mu} := \mu_2 \quad \text{and} \quad \tilde{\nu}_n(\epsilon) := (1 - \epsilon)\nu_{2,n} + \epsilon\kappa$$

where  $\epsilon \in [0, 1]$  and  $\kappa := \frac{1}{2}(\delta_{(-1,0)} + \delta_{(4,0)})$ . We first sketch how we can build a sequence from these measures that violates (S2) and then we look at the irreducible components. Since  $\tilde{\mu} \leq_c \kappa$ , for every sequence  $(\epsilon_n)_{n \in \mathbb{N}_{\geq 1}}$  with  $\epsilon_n \in [0, 1]$ , we have  $\tilde{\mu}_n \leq_c \tilde{\nu}_n(\epsilon_n)$  for all  $n \in \mathbb{N}$ . Moreover if  $(\epsilon_n)_{n \in \mathbb{N}_{\geq 1}}$  tends to zero we have also  $\mathcal{W}_1(\tilde{\nu}_n(\epsilon_n), \mu_2 P_0) \rightarrow 0$  for  $n \rightarrow \infty$ . Note, for  $n \geq 2$  fixed, a martingale transport  $\pi_n(\epsilon)$  between  $\tilde{\mu} = \mu_2$  and  $\tilde{\nu}_n(\epsilon)$  parameterized by  $\epsilon$  must converge to the unique martingale optimal transport between  $\tilde{\mu}$  and  $\nu_{2,n}$  when  $\epsilon \rightarrow 0$  (this is a consequence of Prokhorov's theorem). Paradoxically, this observation amounts in a stability result that allows us to construct a non-stable sequence: for every  $n \geq 1$  we can set  $\epsilon_n$  small enough to satisfy  $|\mathcal{M}_1(\mu_2, \nu_{2,n}) - \mathcal{M}_1(\tilde{\mu}, \tilde{\nu}_n(\epsilon_n))| \leq 1/n$  (and  $\epsilon_n \leq 1/n$ ) and therefore we obtain with Proposition 1.2(ii)

$$\lim_{n \rightarrow \infty} \mathcal{M}_1(\tilde{\mu}, \tilde{\nu}_n(\epsilon_n)) = \lim_{n \rightarrow \infty} \mathcal{M}_1(\mu_2, \nu_{2,n}) > \mathcal{M}_1(\mu_2, \mu_2 P_0) = \mathcal{M}_1(\tilde{\mu}, \mu_2 P_0).$$

Hence, we have a further example that shows that (S2) is in general not satisfied in dimension  $d \geq 2$ . Next, it can be proved with a graphic representation of the measures and a few geometric observations that if  $\pi_n$  is any martingale transport plan between  $\tilde{\mu}$  and  $\tilde{\nu}_n(\epsilon_n)$  the convex hull  $C_1$  of the mass coming from the point  $(1, 0)$  intersects in a two dimensional set the convex hull  $C_2$  of the mass coming

from the point  $(2, 0)$ . As a consequence there is a unique irreducible component for  $\pi_n$  (or the pair  $(\tilde{\mu}, \tilde{\nu}_n(\epsilon_n))$ ) for all definitions of this concept in [10, 8, 20].

**Remark 3.3.** Finally, we would like to point out that Propositions 1.2 and 1.3 and their proofs are not depending on the choice of the Euclidean norm while defining  $c_1$ .

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