21/10/21

(Armand)

Rappel du but du problème de Monge :

Minimiser

$$C : T \in Tr(\mu, \nu) \longmapsto \int_E c(x, T(x))\, \mu(dx)$$

Problème de Monge-Kantorovich

Mêmes données que problème de Monge

1.) E, F Polonais

2.) $\mu \in \mathcal{P}(E)$, $\nu \in \mathcal{P}(E)$ probas

3.) $c : E \times F \to [0, +\infty]$ mesurable

Notation : $\mathrm{Marg}(\mu, \nu) = \{ \pi \in \mathcal{P}(E \times F), \ p_{1\#}\pi = \mu \text{ et } p_{2\#}\pi = \nu \}$

$\mathrm{Marg}(\mu, \nu) =$ ens. plans de transport de μ à ν

$(p_{1\#}\pi = \mu \quad \text{et} \quad p_{2\#}\pi = \nu)$

$$\pi \in \mathrm{Marg}(\mu, \nu) \iff \begin{cases} \forall A \subset E, \ \mu(A) = \pi(A \times F) \\ \forall B \subset F, \ \nu(B) = \pi(E \times B) \end{cases}$$

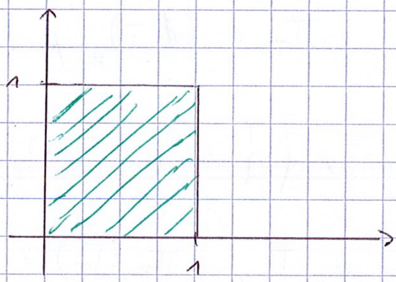
$\mathcal{C}^0_b = \{\text{continues bornées}\}$

$$\iff \begin{cases} \forall f \in \mathcal{C}^0_b(E), \ \int_E f(x)\, d\mu(x) = \int_{E \times F} f(x)\, \pi(dx, dy) \\ \forall g \in \mathcal{C}^0_b(F), \ \int_F g(y)\, \nu(dy) = \int_{E \times F} g(x)\, \pi(dx, dy) \end{cases}$$

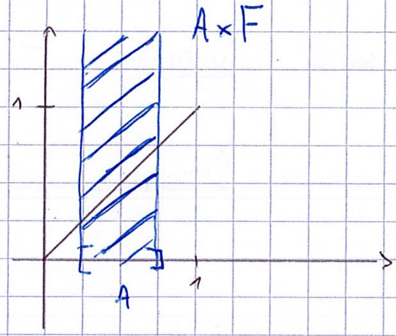
But : Minimiser $J : \underset{\mathrm{Marg}(\mu,\nu)}{\pi \in} \longmapsto \int_{E \times F} c\, d\pi$

Notations : $\mathcal{T}_c(\mu, \nu) = \inf_{\pi \in \mathrm{Marg}(\mu,\nu)} J(\pi)$; $\mathcal{O}_c(\mu, \nu) =$ plans de transport optimaux $\subset \mathrm{Marg}(\mu, \nu)$

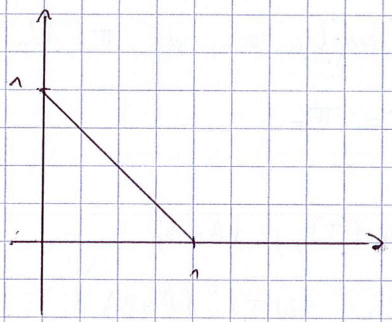
Exemple : $\mu = \nu = \mathcal{U}([0,1])$ (uniforme)



$\pi_1 = \mu \otimes \nu$



$\pi_2 = (id, id)_\# \mu$



$\pi_3 = (id, 1-id)_\# \mu$

Calcul de J pour π_3 :

$$J(\pi_3) = \int c \, d\pi_3 = \int_{\mathbb{R}^2} |x-y| \, (id, 1-id)_\# \mu \, (dx, dy)$$

$$= \int_{\mathbb{R}^2} |x-(1-x)| \, d\mu(x)$$

$$= \int_0^1 |2x-1| \, \mu(dx) = \int_0^{1/2} (1-2x) \, dx + \int_{1/2}^1 (2x-1) \, dx$$

$$= \frac{1}{2}$$

Exercice : calculer $J(\pi_1)$

Lien entre le problème de Monge et celui de M-Kantorovitch

μ mesure sur E

<u>Notation</u> : $T : E \to F$, $\qquad \pi_T = (id, T)_{\#}\, \mu$.

<u>Remarques</u> :

(1) $\pi_T(A \times B) = \mu\left(\left\{ x \in A,\ T(x) \in B \right\}\right)$

(2) $\Gamma_T =$ graphe de T ; $\quad \pi_T(\Gamma_T) = 1$

(3) Si $\pi \in \mathrm{Marg}(\mu, \nu)$, et $\pi(\Gamma_T) = 1$,

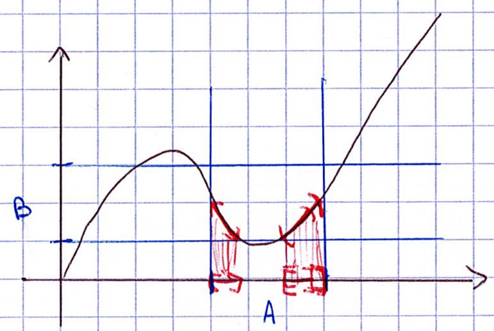
alors $\pi = \pi_T$.

<u>Preuve de (1)</u> : $\pi_T(A \times B) = (id, T)_{\#}\,\mu\,(A \times B)$

$$= \mu\left((id, T)^{-1}(A \times B)\right)$$

$$= \mu\left(\left\{x \in E,\ (x, Tx) \in A \times B\right\}\right)$$

$$= \mu\left(\left\{ x \in A,\ T(x) \in B\right\}\right).$$



<u>Preuve de (3)</u> : Soit $f \in \mathcal{C}_b^0(E \times F)$

$$\int_{E \times F} f \, d\pi = \int_{E \times F} 1\!\!1_{\{(x,y) \in \Gamma_T\}}\, f(x,y)\, \pi(dx, dy)$$

$$= \int_{E \times F} f(x, Tx)\, \pi(dx, dy)$$

$$= \int_E f(x, Tx)\, \mu(dx) = \int_{E\times F} f \,\Big((id, T)_\# \mu \Big) (dx, dy)$$

$$= \int_{E\times F} f \, d\pi_T \,,$$ donc $\pi = \pi_T$ par un résultat de théorie de la mesure.

$$\boxed{\text{\underline{Lemme}} : \quad \pi_T \in \mathrm{Marg}\,(\mu, T_\# \mu)}$$

$$p_{1\#} \pi_T = p_{1\#} \Big((id, T)_\# \mu \Big)$$

$$= \Big(p_1 \circ (id, T) \Big)_\# \mu$$

$$= id_\# \mu = \mu .$$

D'autre part, $\quad p_{2\#} \pi_T = \Big(p_2 \circ (id, T) \Big)_\# \mu = T_\# \mu \quad \blacksquare$

<u>Conséquence</u> : $\quad T \in Tr(\mu, \nu) \iff \pi_T \in \mathrm{Marg}\,(\mu, \nu)$

$$\varphi : T \in Tr(\mu, \nu) \longmapsto \pi_T \in \mathrm{Marg}\,(\mu, \nu)$$

Ainsi $\varphi(T)$ est un plan de transport "de type Monge".

$$J(\pi_T) = \int c \, d\pi_T = \int c \, d\Big((id, T)_\# \mu \Big)$$

$$= \int c \circ (id, T) \, d\mu$$

$$= \int c(x, Tx)\, \mu(dx) = \underset{\substack{\uparrow \\ \text{coût de la séance précédente}}}{C(T)}$$

→ Ainsi, pour résoudre le pb de Monge, on résout celui de M-Kantorovitch restreint aux plans de transport de type Monge.

Cas fini : $E = \{x_1, \dots, x_m\}$ $F = \{y_1, \dots, y_n\}$

- $c : E \times F \to [0, +\infty]$
- $\mu = \sum_{i=1}^{m} p_i \delta_{x_i}$ proba sur E
- $\nu = \sum_{j=1}^{n} q_j \delta_{y_j}$ proba sur F
- $\tilde{C} = \left(c(x_i, y_j)_{i,j} \right)$ (matrice $m \times n$)

Notations : soit $M \in \mathcal{M}_{m \times n}(\mathbb{R}_+)$

(i) $\pi_M = \sum_{i,j} m_{ij} \delta_{(x_i, y_j)}$

(ii) si $\pi \in \mathcal{M}_+(E \times F)$, $\Theta(M) = \left(\pi(x_i, y_j) \right)_{i,j}$

(mesure positive sur $E \times F$)

Remarque : $\pi = \pi_{\Theta(\pi)}$

$\hookrightarrow$ $\pi_{\Theta(\pi)} = \sum_{i,j} \pi(x_i, y_j) \delta_{(x_i, y_j)}$

$\hookrightarrow$ $\Theta(\pi_M) = M$

$P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$

$\mathbb{1}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

de taille n

On note $U(p, q) = \left\{ M \in M_{m,n}(\mathbb{R}_+) \ ; \ M \mathbb{1}_n = p \text{ et } {}^t M \mathbb{1}_m = q \right\}$

Proposition : Θ est une bijection de $\mathrm{Marg}(\mu, \nu)$ vers $U(p, q)$.

$\hookrightarrow$ $M \in \mathcal{M}_{m \times n}(\mathbb{R}_+)$

(i) $\pi_M \in \mathrm{Marg}(\mu, \nu) \iff \begin{cases} \forall A \subset E, \ \pi_M(A \times F) = \mu(A) \\ \forall B \subset F, \ \pi_M(E \times B) = \nu(B) \end{cases}$

$$\iff \begin{cases} \forall i, & \Pi_M(\{x_i\} \times F) = \mu(x_i) \\ \forall j, & \Pi_M(E \times \{y_j\}) = \nu(y_j) \end{cases}$$

$$\iff \begin{cases} \forall i, & \sum_j \Pi_M(x_i, y_j) = \mu(x_i) = p_i \\ \forall j, & \sum_i \Pi_M(x_i, y_j) = \nu(y_j) = q_j \end{cases}$$

("carré magique")

$$\iff M \in U(p,q)$$

(ii) $\tilde{\Theta}(M) := \Pi_M \in \mathrm{Marg}(\mu, \nu)$, $M \in U(p,q)$

(iii) Si $\pi = \pi_{\Theta(\pi)} \in \mathrm{Marg}(\mu,\nu)$, ~~$\Theta(\pi) \in \mathrm{Marg}$~~ $\Theta(\pi) \in U(p,q)$

(iv) $\pi = \tilde{\Theta}(\Theta(\pi))$ et $M = \Theta(\Pi_M) = \Theta(\tilde{\Theta}(M))$ ■

$J : \pi \longmapsto \int c \, d\pi$ $\qquad J(\pi) = \langle \tilde{c}, \Theta(\pi) \rangle$

$$= \sum_{i,j} c_{i,j} \, \Theta(\pi)_{i,j}.$$

<u>Remarque</u> : $\Theta^{-1} = \tilde{\Theta}$. $\qquad \hookrightarrow \langle A, B \rangle = \mathrm{Tr}({}^t A B)$

<u>Remarques</u> :
- $U(p,q)$ est compact, convexe
- $\langle \tilde{c}, \cdot \rangle$ est continue (topologie usuelle de $M_n(\mathbb{R})$)
- Existence d'un minimiseur pour M-Kantorovitch discret garantie (fct° continue sur compact).

$\hookrightarrow$ Cas général plus tard.